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ON PROJECTIVELY FLAT FINSLER SPACES WITH (α, β)-METRIC

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ABSTRACT. The (α, β)-metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β; it has been sometimes treated in theoretical physics. The condition for a Finsler space with an (α, β)-metric $L(\alpha, \beta)$ to be projectively flat was given by Matsumoto [11]. The present paper is devoted to studying the condition for a Finsler space with $L = \alpha^{1-r(x)}\beta^{r(x)}$ or $L = \alpha + \beta^2/\alpha$ to be projectively flat on the basis of Matsumoto's results.

1. Introduction

Let $F^n = (M^n, L)$ be an n-dimensional Finsler space, that is, an n-dimensional differential manifold M^n equipped with a fundamental function $L(x, y)$. The concept of an (α, β)-metric $L(\alpha, \beta)$ was introduced in 1972 by M. Matsumoto [13]. A Finsler metric $L(x, y)$ is called an *(α, β)-metric* $L(\alpha, \beta)$ if L is a positively homogeneous function of α and β of degree one, where $\alpha^2 = a_{ij}(x)y^iy^j$ is a Riemannian metric and $\beta = b_i(x)y^i$ is a one form on M^n. We have specially interesting examples of an (α, β)-metric $L(\alpha, \beta)$. In particular, a Finsler fundamental function $L(x, y)$ will be called the generalized Kropina metric if $L(x, y)$is defined by $L(x, y) = \alpha^m\beta^{1-m}$, where $m \neq 0, 1$. And the Kropina metric was first introduced by L. Berwald in connection with a two-dimensional Finsler space with rectilinear extremal and was investigated by V. K. Kropina.

A Finsler space is called *projectively flat*, or with *rectilinear geodesic*, if the space is covered by coordinate neighborhoods in which the geodesics can be represented by $(n-1)$ linear equations of the coordinates. Such a coordinate system is called *rectilinear*. The condition for a Finsler space

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to be projectively flat was studied by L. Berwald [2] in tensorial form and completed by M. Matsumoto [9]. Hashiguchi and Ichijyō's paper [4] gives interesting results on projective flatness of the Randers spaces.

We have two essential projective invariant tensors, one of them is the *Weyl tensor* W and the other is the *Douglas tensor* D. A Finsler space where both of these tensors vanish is characterized as a projectively flat Finsler space which can be projectively mapped to a locally Minkowski space. The examples of a Finsler space with (α,β)-metric $L(\alpha,\beta)$ are a Randers space, a Kropina space and a special generalized Kropina space with $L=\beta^2/\alpha$. The conditions for the above spaces to be projectively flat were shown by M. Matsumoto [11].

The purpose of the present paper is devoted to studying the condition that a Finsler space with $L=\alpha^{1-r(x)}\beta^{r(x)}$ or $L=\alpha+\beta^2/\alpha$ is projectively flat.

Throughout the present paper we use the terminology and notations in Matsumoto's monograph [10].

2. Preliminaries

We consider a Finsler space with an (α,β)-metric $L(\alpha,\beta)$. First, we are concerned with the *associated Riemannian space* with metric $\alpha(x,y)=\sqrt{a_{ij}(x)y^iy^j}$. Let $\gamma_j{}^i{}_k(x)$ be the Christoffel symbols constructed from α and denoted by (;) the covariant differentiation with respect to $\gamma_j{}^i{}_k$. From the differential 1-form $\beta(x,y)=b_i(x)y^i$ we define

$$2r_{ij}=b_{i;j}+b_{j;i},\quad 2s_{ij}=b_{i;j}-b_{j;i}=(\partial_jb_i-\partial_ib_j),$$
$$s^i{}_j=a^{ir}s_{rj},\quad b^i=a^{ir}b_r,\quad b^2=a^{rs}b_rb_s.$$

Next, we consider the Berwald connection $B\Gamma=(G_j{}^i{}_k,G^i_j,0)$ of the Finsler space with the (α,β)-metric $L(\alpha,\beta)$. As is well-known, we have

$$G_j{}^i{}_k=\dot\partial_kG^i_j,\quad 2G^i=G^i{}_0(=G^i{}_ry^r),\quad G^i{}_j=\dot\partial_jG^i.$$

In the following we denote by the subscript 0 the transvection by y^i and by subscripts α and β of L the partial differentiations by α and β, respectively.

If we put $2B^i = 2G^i - \gamma_0{}^i{}_0$, then the equation (2.5) of a paper [12] gives

$$B^i = (E/\alpha)y^i + (\alpha L_\beta / L_\alpha)s^i_0 - (\alpha L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(y^i/\alpha - \alpha b^i/\beta), \tag{2.1}$$

where quantities E and C are given by

$$C + (\alpha^2 L_\beta/\beta L_\alpha)s_0 + (\alpha L_{\alpha\alpha}/\beta^2 L_\alpha)(\alpha^2 b^2 - \beta^2)(C + \alpha r_{00}/2\beta) = 0, \tag{2.2}$$

$$(2L/\alpha)E = (2\beta L_\beta/\alpha)C + L_\beta r_{00}. \tag{2.3}$$

Now it is well-known [4] that a Finsler space is projectively flat if and only if the space is covered by rectilinear coordinate neighborhoods, that is, in which the G^i is proportional to y^i. Thus we shall quote the following theorem as follows:

THEOREM 2.1. [11]. *A Finsler space with (α, β)-metric $L(\alpha, \beta)$ is projectively flat if and only if the space is covered by coordinate neighborhoods in which the following equation is satisfied:*

$$(\gamma_0{}^i{}_0 - \gamma_{000}y^i/\alpha^2)/2 + (\alpha L_\beta/L_\alpha)s^i_0 + (L_{\alpha\alpha}/L_\alpha)(C + \alpha r_{00}/2\beta)(\alpha^2 b^i/\beta - y^i) = 0. \tag{2.4}$$

Here we shall state the following lemma for the later use [5]:

LEMMA 2.2. *If $\alpha^2 \equiv 0$ (mod.β), that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as a factor, then the dimension is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

We shall state one more remark: Throughout the paper, we shall say *"homogeneous polynomial(s) in (y^i) of degree r"* as $hp(r)$ for brevity. Thus $\gamma_0{}^i{}_0$ are $hp(2)$ and if the space is projective, then py^i are $hp(2)$.

3. Projectively flat Finsler space with $L = \alpha^{1-r(x)}\beta^{r(x)}$.

G. Yu. Bogoslovsky [3] introduced the locally anisotropic space-time with the metric

$$ds = \{(v_i dx^i)^2 / a_{ij}dx^i dx^j\}^{\frac{r}{2}} (a_{ij}dx^i dx^j)^{\frac{1}{2}} \tag{3.1}$$

where $a_{ij}(x)$ is a Riemannian metric and $r = r(x)$. The geometry of the Finsler space with metric (3.1) depends on two additional fields; a scalar field $r(x)$, which determines the magnitude of local anisotropy, and v_i which indicates the locally preferred directions in space-time. Since $(a_{ij}dx^i dx^j)^{\frac{1}{2}}$ is a Riemannian metric and $v_i dx^i$ is a differential one-form, the metric given by (3.1) is considered as an (α, β)-metric $L = \alpha^{1-r(x)}\beta^{r(x)}$ with the function of positive $r(x)$ from the viewpoint of Finsler geometry.

In this section we are to consider the Finsler space equipped with this structure from a geometrical view-point and shall study the Finsler space $F^n = (M^n, L(x,y))$ with the fundamental function $L(x,y)$ given by

$$L = \alpha^{1-r(x)}\beta^{r(x)}, \tag{3.2}$$

where $r(x) \neq -1$, 0, 1 is supposed. Then

$$\begin{aligned} L_\alpha &= (1-r(x))\left(\frac{\beta}{\alpha}\right)^{r(x)}, \quad L_\beta = r(x)\left(\frac{\beta}{\alpha}\right)^{r(x)-1}, \\ L_{\alpha\alpha} &= r(x)(r(x)-1)\frac{\beta^{r(x)}}{\alpha^{r(x)+1}}. \end{aligned} \tag{3.3}$$

Substituting (3.3) in (2.2), we obtain

$$\begin{aligned} &2(1-r(x))\{(1+r(x))\beta^2 - r(x)b^2\alpha^2\}(C + \alpha r_{00}/2\beta) \\ =&(1-r(x))\alpha\beta r_{00} - 2r(x)\alpha^3 s_0. \end{aligned}$$

Suppose that $(1+r(x))\beta^2 - r(x)b^2\alpha^2 = 0$. Then transvection by b^ib^j leads to the contradiction $b^2 = 0$. As $b^2 \neq 0$ is supposed, Lemma shows $\alpha^2 \not\equiv 0$ (mod. β). In this section, we are to treat only the case of $\alpha^2 \not\equiv 0$ (mod. β). Since $(1+r(x))\beta^2 - r(x)b^2\alpha^2$ does not vanish, we have

$$C + \alpha r_{00}/2\beta = \frac{\alpha\{(1-r(x))\beta r_{00} - 2r(x)\alpha^2 s_0\}}{2(1-r(x))\{(1+r(x))\beta^2 - r(x)b^2\alpha^2\}}. \tag{3.4}$$

Substituting (3.3) and (3.4) in (2.4), we get

$$\begin{aligned} &\{(1+r(x))\beta^2 - r(x)b^2\alpha^2\}\{(1-r(x))\beta(\alpha^2\gamma_0{}^i{}_0 - \gamma_{000}y^i) \\ &+ 2r(x)\alpha^4 s^i_0\} - r(x)\alpha^2\{(1-r(x))\beta r_{00} - 2r(x)\alpha^2 s_0\} \\ &(\alpha^2 b^i - \beta y^i) = 0. \end{aligned} \tag{3.5}$$

First, the term in (3.5) which seemingly does not contain β is $2r(x)^2\alpha^6$ $(s_0 b^i - b^2 s^i_0)$ only. Hence we must have function $\lambda^i(x)$ satisfying

$$s_0 b^i - b^2 s^i_0 = \beta\lambda^i. \tag{3.6}$$

The above equation (3.6) is written in the form

$$b^2 s_{ik} = b_i s_k - b_k \lambda_i; \qquad \lambda_i = a_{ik}\lambda^k. \tag{3.7}$$

Transvecting (3.7) by b^k and referring to skew-symmetric s_{ij}, we have $s_i = \lambda_i$. Thus the equation (3.7) is written as

$$b^2 s_{ij} = b_i s_j - b_j s_i. \tag{3.8}$$

Secondly, we observe in (3.5) that the term $(r(x)^2 - 1)\beta^3\gamma_{000}y^i$ must have a factor α^2. Then the method which has been applied to (3.6) leads us to the existence of $v_0 = v_i(x)y^i$ satisfying

$$\gamma_{000} = \alpha^2 v_0. \tag{3.9}$$

Substituting from (3.8) and (3.9), the equation (3.5) reduces to

$$\begin{aligned}
&\alpha^2\big[r(x)(r(x)-1)b^2({\gamma_0}^i{}_0 - v_0 y^i) + r(x)\{2(r(x)+1)\beta s_0/b^2 \\
&+ (r(x)-1)r_{00}\}b^i + 2r(x)\{r(x)\alpha^2 - (r(x)+1)\beta^2/b^2\}s^i \\
&- 2r(x)^2 s_0 y^i\big] \\
&= (r(x)-1)\beta\{(r(x)+1)\beta({\gamma_0}^i{}_0 - v_0 y^i) + r(x)r_{00}y^i\}.
\end{aligned} \tag{3.10}$$

The terms of (3.10) which seemingly do not contain a factor α^2 are

$$(r(x)-1)\beta\{(r(x)+1)\beta({\gamma_0}^i{}_0 - v_0 y^i) + r(x)r_{00}y^i\}.$$

Consequently we must have $\xi^i_0 = \xi^i_j(x)y^j$ satisfying

$$(r(x)+1)\beta({\gamma_0}^i{}_0 - v_0 y^i) + r(x)r_{00}y^i = \alpha^2\xi^i_0. \tag{3.11}$$

Transvecting (3.11) by y^i and making use of (3.9), the equation (3.11) leads us to

$$r(x)r_{00} = \xi^i_o y_i = \xi_{00}. \tag{3.12}$$

Substituting from (3.11), the equation (3.10) reduces to

$$
\begin{aligned}
&r(x)(r(x)-1)b^2(\gamma_0{}^i{}_0 - v_0 y^i) + r(x)\{2(r(x)+1)\beta s_0/b^2 \\
(3.13)\qquad &+ (r(x)-1)r_{00}\}b^i + 2r(x)\{r(x)\alpha^2 - (r(x)+1)\beta^2/b^2\}s^i \\
&- 2r(x)^2 s_0 y^i = (r(x)-1)\beta\xi^i_0.
\end{aligned}
$$

Multiplying (3.13) by $(r(x)+1)\beta$, where $r(x) \neq -1$ is supposed, we have

$$
\begin{aligned}
&r(x)(r(x)-1)(r(x)+1)b^2\beta(\gamma_0{}^i{}_0 - v_0 y^i) \\
&+ r(x)(r(x)+1)\beta\{2(r(x)+1)\beta s_0/b^2 + (r(x)-1)r_{00}\}b^i \\
(3.14)\qquad &+ 2r(x)(r(x)+1)\beta\{r(x)\alpha^2 - (r(x)+1)\beta^2/b^2\}s^i \\
&- 2(r(x)+1)r(x)^2\beta s_0 y^i = (r(x)-1)(r(x)+1)\beta^2\xi^i_0.
\end{aligned}
$$

Substituting from (3.11) and (3.12), we are led to

$$
\begin{aligned}
&\{r(x)(1-r(x))b^4\alpha^2\xi_{i0} - (1-r(x))(r(x)+1)b^2\beta^2\xi_{i0} \\
&- 2r(x)^2(r(x)+1)b^2\alpha^2\beta s_i + 2r(x)(r(x)+1)^2\beta^3 s_i\} \\
&- \{r(x)(1-r(x))b^4\xi_{00}y_i - (r(x)+1)(1-r(x))b^2\beta\xi_{00}b_i \\
&- 2r(x)^2(r(x)+1)b^2\beta s_0 y_i + 2r(x)(r(x)+1)^2\beta^2 s_0 b_i\} = 0.
\end{aligned}
$$

Thus the above equation is rewritten as follows:

$$
\begin{aligned}
&\{r(x)b^2\alpha^2 - (r(x)+1)\beta^2\}\{(r(x)-1)b^2\xi_{i0} + 2r(x) \\
(3.15)\qquad &(r(x)+1)\beta s_i\} = \{(r(x)-1)b^2\xi_{00} + 2r(x)(r(x)+1)\beta s_0\} \\
&\{r(x)b^2 y_i - (r(x)+1)\beta b_i\},
\end{aligned}
$$

where we put $\xi_{ij} = a_{ir}\xi^r_j$.
If we define the tensors

$$
\begin{aligned}
M_{ij} &= r(x)b^2 a_{ij} - (r(x)+1)b_i b_j, \\
N_{ij} &= (r(x)-1)b^2\xi_{ij} + 2r(x)(r(x)+1)s_i b_j,
\end{aligned}
$$

then (3.15) is written in the form

$$
\begin{aligned}
&2M_{hj}N_{ik} + 2M_{jk}N_{ih} + 2M_{kh}N_{ij} \\
(3.16)\qquad &= (N_{jk} + N_{kj})M_{ih} + (N_{kh} + N_{hk})M_{ij} + (N_{hj} + N_{jh})M_{ik}.
\end{aligned}
$$

It is to prove that the tensor M_{ij} obtains the reciprocal M^{ij} as follows:

$$M^{ij} = \{a^{ij} - (r(x)+1)b^i b^j/b^2\}/r(x)b^2.$$

Thus, transvecting (3.16) by M^{hj}, we get

$$(n+1)N_{ik} = N_{ki} + 2NM_{ik}, \quad nN = M^{hj}N_{hj},$$

which gives $N_{ik} = N_{ki} = NM_{ik}$ immediately. Therefore, we obtain

$$\begin{aligned}(r(x)-1)b^2\xi_{ik} =& N\{r(x)b^2 a_{ik} - (r(x)+1)b_i b_k\} \\ & - 2r(x)(r(x)+1)s_i b_k,\end{aligned} \tag{3.17}$$

and (3.12) is rewritten in the form

$$\begin{aligned}r(x)(r(x)-1)b^2 r_{00} =& N(r(x)b^2\alpha^2 - (r(x)+1)\beta^2) \\ & - 2r(x)(r(x)+1)s_0\beta.\end{aligned} \tag{3.18}$$

Substituting from (3.17) and (3.18), the equation (3.13) is rewritten as follows:

$$(r(x)-1)b^2({\gamma_0}^i{}_0 - v_0 y^i) = (2r(x)s_0 + N\beta)y^i - N\alpha^2 b^i - 2r(x)\alpha^2 s^i. \tag{3.19}$$

Conversely, it is easily verified that (3.5) is a consequence of (3.8), (3.18) and (3.19). These equations (3.18) and (3.19) may be written, respectively, in the forms

$$\begin{aligned}r(x)(r(x)-1)b^2 r_{ij} =& N\{r(x)b^2 a_{ij} - (r(x)+1)b_i b_j\} \\ & - r(x)(r(x)+1)(s_i b_j + s_j b_i),\end{aligned} \tag{3.20}$$

$$(r(x)-1)b^2{\gamma_j}^i{}_k = b^2(\delta^i_j\sigma_k + \delta^i_k\sigma_j) - (Nb^i + 2r(x)s^i)a_{jk}, \tag{3.21}$$

where we put $2\sigma_j = (r(x)-1)v_j + (2r(x)s_j + Nb_j)/b^2$.

Thus we have the following

THEOREM 3.1. *An n-dimensional Finsler space F^n with $L = \alpha^{1-r(x)}\beta^{r(x)}$, where $r(x) \neq -1$, 0, 1, is projectively flat, if and only if we have (3.8) and (3.20), and the space is covered by coordinate neighborhoods in which the Christoffel symbols of the associated Riemannian space with the metric α are written in the form (3.21).*

4. Projectively flat Finsler space with $L = \alpha + \beta^2/\alpha$.

The present section is devoted to a Finsler space F^n with the metric

$$L(\alpha, \beta) = \alpha + \beta^2/\alpha. \tag{4.1}$$

This metric is now proposed and is thought of as desirable in the view-point of geometry and of applications. Since α is a Riemannian metric, this metric L is nearly allied to a Riemannian metric. $L(\alpha, \beta)$ may be regarded as constructed from α and one more Riemannian metric $\sqrt{\alpha^2 + \beta^2}$.

From (4.1) we obtain

$$L_\alpha = \frac{\alpha^2 - \beta^2}{\alpha^2}, \quad L_\beta = \frac{2\beta}{\alpha}, \quad L_{\alpha\alpha} = \frac{2\beta^2}{\alpha^3}. \tag{4.2}$$

Substituting (4.3) in (2.2), we have

$$C + \alpha r_{00}/2\beta = \frac{\alpha\{(\alpha^2 - \beta^2)r_{00} - 4\alpha^2\beta s_0\}}{2\beta\{(1 + 2b^2)\alpha^2 - 3\beta^2\}}. \tag{4.3}$$

Here the denominator $2\beta\{(1+2b^2)\alpha^2 - 3\beta^2\}$ does not vanish. In fact, if $2\beta\{(1 + 2b^2)\alpha^2 - 3\beta^2\} = 0$, then $(1 + 2b^2)a_{ij} = 3b_ib_j$, so we have a contradiction.

Substituting (4.2) and (4.3) in (2.4), we get

$$\begin{aligned}&(\alpha^2 - \beta^2)\{(1 + 2b^2)\alpha^2 - 3\beta^2\}(\alpha^2\gamma_0{}^i{}_0 - r_{00}y^i)\\ &+ 4\alpha^4\beta\{(1 + 2b^2)\alpha^2 - 3\beta^2\}s^i_0 + 2\alpha^2\{(\alpha^2 - \beta^2)r_{00}\\ &- 4\alpha^2\beta s_0\}(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{4.4}$$

Only the term $-3\beta^4 r_{000}y^i$ of (4.4) seemingly does not contain α^2 and hence we must have $hp(6)$ v^i_6 satisfying $-3\beta^4 r_{000}y^i = \alpha^2 v^i_6$. For the sake of brevity we suppose $\alpha^2 \not\equiv 0$ (mod. β). Then the above is written as

$$r_{000} = v_0\alpha^2, \tag{4.5}$$

where v_0 is $hp(1)$. Substituting from (4.5), the equation (4.4) reduces to

$$\begin{aligned}&(\alpha^2 - \beta^2)\{(1 + 2b^2)\alpha^2 - 3\beta^2\}(\gamma_0{}^i{}_0 - v_0y^i) + 4\alpha^2\beta\{(1 + 2b^2)\alpha^2\\ &- 3\beta^2\}s^i_0 + 2\{(\alpha^2 - \beta^2)r_{00} - 4\alpha^2\beta s_0\}(\alpha^2 b^i - \beta y^i) = 0.\end{aligned} \tag{4.6}$$

The terms of (4.6) which seemingly do not contain α^2 are $\beta^3\{3\beta(\gamma_0{}^i{}_0 - v_0y^i)+2r_{00}y^i\}$. Consequently we must have $hp(1)$ u_0^i such that the above is equal to $\alpha^2\beta^3u_0^i$. Thus we come by

$$(4.7) \qquad 3\beta(\gamma_0{}^i{}_0 - v_0y^i) + 2r_{00}y^i = \alpha^2u_0^i.$$

Contraction of (4.7) by $a_{ir}y^r$ leads to

$$(4.8) \qquad 2r_{00} = u_0^iy_i.$$

Substituting (4.8) in (4.7), we obtain

$$(4.9) \qquad \gamma_0{}^i{}_0 = v_0y^i,$$

which yields

$$(4.10) \qquad 2\gamma_j{}^i{}_k = v_k\delta_j^i + v_j\delta_k^i.$$

Consequently the equation (4.10) shows that the associated Riemannian space is projectively flat.

Next, substituting (4.9) in (4.6), we have

$$(4.11) \qquad \begin{aligned} &4\alpha^2\beta\{(1+2b^2)\alpha^2 - 3\beta^2\}s_0^i + 2\{(\alpha^2-\beta^2)r_{00} \\ &- 4\alpha^2\beta s_0\}(\alpha^2b^i - \beta y^i) = 0. \end{aligned}$$

Transvection of (4.11) by b_i leads to

$$(4.12) \qquad 2\alpha^2\beta s_0 + (b^2\alpha^2 - \beta^2)r_{00} = 0.$$

Since $b^2\alpha^2 - \beta^2$ of (4.12) does not contain α^2 as a factor, we must have a function $k(x)$ such that

$$(4.13) \qquad r_{00} = k(x)\alpha^2.$$

Substituting form (4.13), the equation (4.12) reduces to $2\beta s_0 + k(x)(b^2\alpha^2 - \beta^2) = 0$, which gives $(b_is_j + b_js_i) + k(x)(b^2a_{ij} - b_ib_j) = 0$. Transvection by y^iy^j leads to $k(x) = 0$, because $b^2\alpha^2 - \beta^2$ does not vanish. Thus we get

$$(4.14) \qquad r_{00} = 0;\ r_{ij} = 0 \quad \text{and} \quad s_0 = 0;\ s_i = 0.$$

Substituting from (4.14), the equation (4.11) reduces to $s_0^i = 0$, that is, $s_{ij} = 0$.

Summarizing up, we obtain $b_{i;j} = 0$ from $r_{ij} = 0$ and $s_{ij} = 0$.

Conversely, if $b_{i;j} = 0$, then we come by $r_{00} = 0$, $s_0^i = 0$ and $s_0 = 0$. So (4.6) follows from (4.9).

Thus we have the following

THEOREM 4.1. *A Finsler space with an (α, β)-metric $L(\alpha, \beta)$ given by (4.1) is projectively flat, if and only if we have $b_{i;j} = 0$ and the associated Riemannian space (M^n, α) is projectively flat.*

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