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RADICALS AND HOMOMORPHIC IMAGES OF C*-ALGEBRAS

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ABSTRACT. In this paper, we prove that the range of homomorphism from a C*-algebra A into a commutative Banach algebra B whose radical is nil contains no non-zero element of the radical of B. Using this result we show that there is no non-zero homomorphism from a C*-algebra into a commutative radical nil Banach algebra.

1. Introduction

Let A and B be Banach algebras. A linear map $\theta : A \to B$ from A into B is said to be a homomorphism if θ is multiplicative. There are various fruitful results in continuity of homomorphisms between Banach algebras. But the existence problem of non-zero homomorphisms between Banach algebras has scarcely been studied so far.

In this paper, we study the existence problem of a non-zero homomorphism from a C*-algebra into a Banach algebra. It is shown that the range of homomorphism from a C*-algebra A into a commutative Banach algebra B whose radical is nil contains no non-zero element of the radical of B. Using this result we show that there is no non-zero homomorphism from a C*-algebra into a commutative radical nil Banach algebra. Finally we give an example of a commutative radical nil Banach algebra and we prove that this algebra can not be a homomorphic image of a C*-algebra.

2. Preliminaries

In this section, we present some definitions and theorems which will be used in section 3.

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DEFINITION 1. The (Jacobson) *radical* of a Banach algebra A is the intersection of the maximal modular left ideals of A if such ideals exist, and is the algebra A itself if there are no maximal modular left ideals of A. The radical of A is denoted by $\mathrm{rad}(A)$. The Banach algebra A is said to be *semi-simple* if $\mathrm{rad}(A)$ contains only the zero element of A, and to be a *radical algebra* if $\mathrm{rad}(A)$ is A itself.

By the definition of the radical of a Banach algebras, the radical is a closed left ideal, but it is in fact a closed two-sided ideal of the algebra.

An element x of a C*-algebra is called *hermitian* if $x = x^*$, and a subalgebra is called *self-adjoint* if it is closed under involution. Every closed two-sided ideal of a C*-algebra is known to be self-adjoint.

An element x in a Banach algebra is said to be *quasi-nilpotent* if $\|x^n\|^{\frac{1}{n}} = 0$. Since $\|x^n\|^{\frac{1}{n}}$ is the spectral radius of x, an element x of a Banach algebra is quasi-nilpotent if and only if the spectral radius of x is equal to zero. That is the spectrum of x contains a single element 0. In a Banach algebra every element of the radical is quasi-nilpotent. But in a noncommutative Banach algebra a quasi-nilpotent element need not be in the radical. In fact, Kaplansky proved that every noncommutative C*-algebra contains a non-zero nilpotent element and this element does not belong to the radical since every C*-algebra is semi-simple. In a commutative Banach algebra the radical coincides with the set of quasi-nilpotent elements.

The following theorem is the earliest result in automatic continuity theory. This was proved by Silov and can be found in [2]. But Silov's result does not hold when A is noncommutative [4]. Johnson proved that every epimorphism from a Banach algebra onto a semi-simple Banach algebra is continuous [7]. It is still open question whether every homomorphism from a Banach algebra into a semi-simple Banach algebra with dense range is continuous.

THEOREM 2. *Every homomorphism from a Banach algebra into a commutative semi-simple Banach algebra is continuous.*

The following theorem shows that the range of continuous homomorphism from a C*-algebra A into a Banach algebra B contains no non-zero element of the radical of B. This theorem is found in [3].

THEOREM 3. *Let A be a C*-algebra and B be a Banach algebra. Then for each continuous homomorphism θ from A into B we have*

$$\mathrm{rad}(B) \cap \theta(A) = \{0\}.$$

An ideal I of a algebra A is said to be a *nil ideal* if for each $x \in I$, there exists a positive integer n such that $x^n = 0$, where n depends upon the element $x \in I$. An ideal I of a algebra A is said to be a *nilpotent ideal* if $I^n = \{0\}$ for some positive integer n. Here, I^n denotes the set of all finite sums of product of n elements taken from I. Hence a nilpotent ideal is a nil ideal. But S. Grabiner proved that every nil Banach algebra is nilpotent [2].

The next theorem gives a sufficient condition for a homomorphism of $C(\Omega)$ to be continuous. Here, $C(\Omega)$ denotes the commutative C*-algebra of continuous functions on a compact Hausdorff space Ω. The theorem is found in [1].

THEOREM 4. *Let θ be a homomorphism of $C(\Omega)$ into a commutative Banach algebra B. If the radical of B is a nil ideal, then θ is continuous.*

3. Homomorphic images of C*-algebras

In this section, we investigate the properties of the radical of the codomain of a homomorphism on a C*-algebra.

LEMMA 5. *Let A be a Banach algebra and $A_I = A \oplus \mathbb{C}$ be the unitization of A. Then we have*

$$\mathrm{rad}(A) = \mathrm{rad}(A_I).$$

PROOF. Define a map $f : A_I \to \mathbb{C}$ by $f(x, \lambda) = \lambda$. Then clearly f is a multiplicative linear functional on A_I with the kernel A. Hence A is a maximal modular left ideal of A_I. Hence $\mathrm{rad}(A_I) \subseteq A$. Therefore we have,

$$\mathrm{rad}(A) = \mathrm{rad}(A_I) \cap A = \mathrm{rad}(A_I). \qquad \square$$

THEOREM 6. *Let A be a C^*-algebra and B be a commutative Banach algebra whose radical is nil. Then for each homomorphism θ from A into B we have*

$$rad(B) \cap \theta(A) = \{0\}.$$

PROOF. Let $\theta : A \to B$ be a homomorphism from A into B and let b be an arbitrary element of $\mathrm{rad}(B) \cap \theta(A)$. Then there exists an element a in A with $\theta(a) = b$. Let A' be the C*-algebra generated by the hermitian element $a + a^*$. Let $\pi : B \to B/\mathrm{rad}(B)$ be the canonical quotient map. And let $\overline{\theta} = \pi \circ \theta$. Since $B/\mathrm{rad}(B)$ is a commutative semi-simple Banach algebra, $\overline{\theta} : A \to B/\mathrm{rad}(B)$ is continuous. Hence $\ker(\overline{\theta})$, the kernel of $\overline{\theta}$, is a closed two-sided ideal of A and so it is self-adjoint. Since $a \in \ker(\overline{\theta})$, $a^* \in \ker(\overline{\theta})$. That is $\theta(a^*) \in \mathrm{rad}(B)$. Therefore we have,

$$\theta(a + a^*) = b + \theta(a^*) \in \mathrm{rad}(B).$$

Let $\theta' : A' \to B$ be the restriction of θ on A'. Let A'_I, B_I be the unitization of A', B, respectively. And let $j : A' \to A'_I$, $k : B \to B_I$ be the inclusion maps. Let Φ be the carrier space of the commutative C*-algebra A'_I. Then the Gelfand representation $\Lambda : A'_I \to C(\Phi)$, $\Lambda(x, \lambda) = (x, \lambda)\hat{}$ $(x \in A', \lambda \in \mathbb{C})$, is an isometric *-isomorphism of A'_I onto $C(\Phi)$. Define a map $\psi : C(\Phi) \to B_I$ by $\psi((x, \lambda)\hat{}) = (\theta'(x), \lambda)$. Then clearly ψ is a homomorphism of $C(\Phi)$ into B_I. And we have

$$\psi \circ \Lambda \circ j = k \circ \theta'.$$

By Lemma 5, $\mathrm{rad}(B_I) = \mathrm{rad}(B)$. Hence $\mathrm{rad}(B_I)$ is a nil ideal. By Theorem 4, $\psi : C(\Phi) \to B_I$ is continuous. So $k \circ \theta'$ is continuous. Since k is the inclusion map, θ' is continuous. By Theorem 3, we have

$$\mathrm{rad}(B) \cap \theta'(A') = \{0\}.$$

Since θ' is the restriction of θ on A', $b + \theta(a^*) \in \mathrm{rad}(B) \cap \theta'(A')$. Therefore we have,

$$b + \theta(a^*) = 0.$$

Similarly considering the C*-subalgebra generated by the hermitian element $i(a - a^*)$, we can show that

$$b - \theta(a^*) = 0.$$

Therefore,

$$b = \frac{1}{2}\{(b + \theta(a^*)) + (b - \theta(a^*)\} = 0.$$

This completes the proof. □

If B is a Banach algebra satisfying the descending chain condition for left (or right) ideals, then the radical of B is nilpotent. Consequently we have the following corollary from Theorem 6.

COROLLARY 7. *Let A be a C^*-algebra and B be a commutative Banach algebra satisfying the descending chain condition for left ideals. Then for each homomorphism θ from A into B we have*

$$\text{rad}(B) \cap \theta(A) = \{0\}.$$

COROLLARY 8. *Let A be a C^*-algebra and B be a commutative radical nil Banach algebra. If $\theta : A \to B$ is a homomorphism from A into B then $\theta = 0$.*

PROOF. Since $\text{rad}(B) \cap \theta(A) = \{0\}$ and $\text{rad}(B) = B$, $\theta = 0$. □

COROLLARY 9. *Let B be a commutative non semi-simple Banach algebra whose radical is nil. Then B is not a homomorphic image of a C^*-algebra.*

PROOF. If there is a homomorphism from a C*-algebra A onto B, then we have

$$\text{rad}(B) = \text{rad}(B) \cap \theta(A) = \{0\}.$$

Hence B must be semi-simple. □

EXAMPLE 10. Let $L^1[0,1]$ denote the Banach space of complex valued integrable functions on $[0,1]$. Here, the norm is defined by

$$\|f\| = \int_0^1 |f(t)|dt \quad (f \in L^1[0,1]).$$

If we define

$$(f * g)(t) = \int_0^t f(s-t)g(s)ds \quad (t \in [0,1])$$

for f, $g \in L^1[0,1]$, then $L^1[0,1]$ is a commutative Banach algebra without identity. This algebra is called the *Volterra algebra* and denoted by V. If $u(t) = 1$ $(t \in [0,1])$, then

$$u^{*n}(t) = \frac{t^{n-1}}{(n-1)!} \quad (n \in \mathbb{N},\ t \in [0,1]),$$

where u^{*n} denotes the n-th convolution power of u. Hence u is a quasi-nilpotent element. Since the radical of V coincides with the set of all quasi-nilpotent elements in V, u belongs to the radical of V. Hence n-th convolution power of u belongs to the radical of V. That is, the functions of the form t^n $(t \in [0,1],\ n \in \mathbb{N})$ belong to the radical of V. Hence all (ordinary) polynomials on $[0,1]$ belong to the radical of V. But then all (ordinary) polynomials on $[0,1]$ are dense in V. It follows that V is a radical algebra.

For a non-zero element of f of V, let $\alpha(f) = \inf(\operatorname{supp} f)$, where $\operatorname{supp} f$ is the support of f. Set $\alpha(0) = \infty$. For $0 < a < 1$, let

$$M_a = \{f \in V : \alpha(f) \geq a\}.$$

By the Titchmarsh convolution theorem [5], M_a is a non-zero closed ideal of V, and if $f \in M_a$ then $f^{*n} \in M_{na}$ $(n \in \mathbb{N})$. Hence M_a is a commutative radical nil Banach algebra.

If θ is a homomorphism from a C*-algebra into M_a, then $\theta = 0$ by Corollary 8.

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