Comm. Korean Math. Soc. 14 (1999), No. 2, pp. 353–364

HYPERBOLIC CONVOLUTION EQUATION IN THE BEURLING'S GENERALIZED DISTRIBUTION SPACE

Dae Hyeon Pahk and Byung Keun Sohn

Abstract. We found the characterizations for convolution operators in the Beurling's generalized distribution space to be hyperbolic.

In [4], L. Ehrenpreis defined the hyperbolic convolution operator $S(t,x) \in \mathcal{E}'(R \times R^n)$, the distribution space with compact support, in $t^-(t^+)$ and showed that S is hyperbolic in $t^-(t^+)$ if and only if there exists $C > 0$ such that

$$Im\tau \geq -C(1 + |Imz| + \log(1 + |\tau| + |z|))$$

$$(Im\tau \leq C(1 + |Imz| + \log(1 + |\tau| + |z|))),$$

for $(\tau, z) \in V = \{(\tau, z) \in C \times C^n; \widehat{S}(\tau, z) = 0\}$.

Later, C. C. Chou [3] extended this result to $S(t,x) \in \mathcal{E}'(M_{(p)}, R \times R^n)$, the Romieu's ultradistribution space with compact support, by showing that S is $M_{(p)}$- hyperbolic in $t^-(t^+)$ if and only if there exist positive constants a and H such that

$$Im\tau \geq -a(H|Imz| + M(\tau, z))$$

$$(Im\tau \leq a(H|Imz| + M(\tau, z))),$$

Received October 9, 1998. Revised January 19, 1999.
1991 Mathematics Subject Classification: 44A35, 46F10.
Key words and phrases: hyperbolic convolution equation, Beurling's distribution.
The first author was supported by Yonsei Research Fund '98, BSRI 97-1421 and GARC and the second author was supported by Inje Research and Scholarship Foundation, 1997.

for $(\tau, z) \in V$. Here $M(\tau, z)$ is the associate function to a sequence $\{M_p\}$. Also, in [5] D. H. Pahk and B. H. Kang studied the hyperbolic differential equation in the Beurling's generalized distribution space. In this paper, we extend the L. Ehrenpreis and D. H. Pahk and B. H. Kang's results to hyperbolic convolution equation in the Beurling's generalized distribution space.

For the completeness, we briefly review the Beurling's generalized distribution space and related the results which we need in this paper. For details, we refer to [2]. We denote $\mathcal{M}_c$ the set of all real-valued functions ω on R^n satisfying the following conditions;

(α) $0 = \omega(0) \leq \omega(\xi + \eta) \leq \omega(\xi) + \omega(\eta), \xi, \eta \in R^n$

(β) $\displaystyle\int_{R^n} \frac{\omega(\xi)}{(1 + |\xi|)^{n+1}} d\xi < \infty$

(γ) $\omega(\xi) \geq a + b\log(1 + |\xi|)$ for some constants a and $b > 0$

(δ) $\omega(\xi) = \sigma(|\xi|)$ for an increasing concave function σ on $[0, \infty)$.

For example, $\omega(\xi) = \log(1 + |\xi|)$ and $\omega(\xi) = |\xi|^{\frac{1}{\alpha}}, \alpha > 1$, satisfy all conditions. Throughout this paper, ω represents an element in $\mathcal{M}_c$. Let $\mathcal{D}_\omega(U)$ be the set of all ϕ in $L^1(R^n)$ such that ϕ has a compact support in an open set U and

$$\| \phi \|_\lambda^{(\omega)} = \int_{R^n} |\widehat{\phi}(\xi)| e^{\lambda\omega(\xi)} d\xi < \infty \text{ for any } \lambda > 0.$$

The topology on this space is given by the inductive limit topology of the Fréchet spaces $\mathcal{D}_\omega(K) = \{\phi \in \mathcal{D}_\omega; \operatorname{supp}\phi \subset K\}$ induced by the above semi-norms where K is a compact set in U. We denote by $\mathcal{E}_\omega(U)$ the set of all complex-valued functions ψ in U such that $\phi\psi$ is in $\mathcal{D}_\omega(U)$ for any $\phi \in \mathcal{D}_\omega(U)$. The topology in $\mathcal{E}_\omega(U)$ is given by the semi-norms $\psi \to \| \phi\psi \|_\lambda^{(\omega)}$ for any $\lambda > 0$ and any $\phi \in \mathcal{D}_\omega(U)$. The dual space of $\mathcal{D}_\omega(U)$ is denoted by $\mathcal{D}'_\omega(U)$ whose elements are called the Beurling's generalized distributions because of $\mathcal{D}'_\omega(U) \supset \mathcal{D}'(U)$ by (γ). The dual space $\mathcal{E}'_\omega(U)$ of $\mathcal{E}_\omega(U)$ can be identified with the set of all elements of $\mathcal{D}'_\omega(U)$ which has a compact support in U. $\mathcal{D}'_\omega(U)$ is equal to $\mathcal{D}'(U)$

when $\omega(\xi) = \log(1 + |\xi|)$ and $\mathcal{E}_\omega(U)$ is related to the Gevery class when $\omega(\xi) = |\xi|^{\frac{1}{d}}$, $d > 1$. If $v * \phi(x) = \langle v_y, \phi(x - y)\rangle$ and $\langle u * v, \phi\rangle = \langle v, \check{u} * \phi\rangle$ for $u \in \mathcal{E}'_\omega$, $v \in \mathcal{D}'_\omega$ and $\phi \in \mathcal{D}_\omega$, it can be easily seen that $\mathcal{D}'_\omega * \mathcal{D}_\omega \subset \mathcal{E}_\omega$ and $\mathcal{E}'_\omega * \mathcal{D}'_\omega \subset \mathcal{D}'_\omega$. Because of the later fact, $\mathcal{E}'_\omega$ is called the space of convolution operators in $\mathcal{D}'_\omega$.

LEMMA 1 [2]. *$\omega(\xi) = O(|\xi|/\log|\xi|)$ where $|\xi| \to \infty$ for all $\omega \in \mathcal{M}_c$.*

It follows from this Lemma 1 that there exists a constant M such that

$$\omega(\xi) \leq M(1 + |\xi|),\ \xi\ \in R^n.$$

LEMMA 2 [2]. *Let $\omega \in \mathcal{M}_C$ and let K be a compact subset of R^n. Then the family of semi-norms $\{\phi \to \|\phi\|_\lambda^{(\omega)}\}_{\lambda>0}$ on $\mathcal{D}_\omega(K)$ is equivalent to the family $\{\phi \to \sup_{\zeta\in C^n}|\widehat{\phi}|e^{(\lambda\omega(\xi)-U_K(\eta)-|\eta|)}\}_{\lambda>0}$, where $\zeta = \xi + i\eta$ and U_K is the supporting function of K, i.e., $U_K(\eta) = \max_{x\in K}\langle x, \eta\rangle$.*

LEMMA 3. *There exists a function $\varphi_\epsilon \in \mathcal{E}_\omega$ whose value is 1 on the set $\{t \in R; |t| < b - \epsilon\} \times R^n$ and 0 on the set $\{t \in R; |t| > b - \frac{\epsilon}{2}\} \times R^n$ for any $b > 0$.*

PROOF. By [5], we can take $\varphi_\epsilon^1 \in \mathcal{E}_\omega$ whose value is 1 on the set $\{(t, x); t < b - \epsilon\} \times R^n$ and 0 on the set $\{(t, x); t > b - \frac{\epsilon}{2}\} \times R^n$. Similarly, we can also take $\varphi_\epsilon^2 \in \mathcal{E}_\omega$ whose value is 1 on the set $\{(t, x); t > -(b - \epsilon)\} \times R^n$ and 0 on the set $\{(t, x); t < -(b - \frac{\epsilon}{2})\} \times R^n$. Then $\varphi_\epsilon = \varphi_\epsilon^1 \cdot \varphi_\epsilon^2 \in \mathcal{E}_\omega$ by the definition of $\mathcal{E}_\omega$ and φ_ϵ satisfies the condition. □

LEMMA 4 [5]. *If $u \in \mathcal{E}'_\omega$ and $\phi \in \mathcal{E}_\omega$, then $u * \phi \in \mathcal{E}_\omega$.*

LEMMA 5 [2]. *Let K be a compact convex set in R^n with a support function H.*

1. The Fourier-Lapalace transform of $\varphi \in \mathcal{D}_\omega(K)$ is an entire function $U(\zeta)$ in $\zeta = \xi + i\eta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in C^n$ if and only if for each λ and for each $\epsilon > 0$ there exists a constant $C_{\lambda,\epsilon}$ such that

$$|U(\xi + i\eta)| \leq C_{\lambda,\epsilon}e^{H(\eta)+\epsilon|\eta|-\lambda\omega(\xi)}.$$

2. *The Fourier-Lapalace transform of* $u \in \mathcal{E}'_\omega(K)$ *is an entire function* $U(\zeta)$ *in* $\zeta = \xi + i\eta = (\zeta_1, \zeta_2, \cdots, \zeta_n) \in C^n$ *if and only if for some real* λ *and all positive* ϵ *there exists a constant* $C_{\lambda,\epsilon}$ *such that*

$$|U(\xi + i\eta| \leq C_{\lambda,\epsilon} e^{H(\eta)+\epsilon|\eta|+\lambda\omega(\xi)}.$$

LEMMA 6 [Levin's Theorem]. *Let* g *be a function of complex variable* ζ *and holomorphic in a neighborhood of* $\{\zeta; |\zeta| \leq 3eR\}$ *and not 0 in* $|\zeta| \leq \frac{3}{2}r$. *Then for all* ζ_0 *with* $|\zeta_0| = R$,

$$|g(0)| \geq \frac{|g(\zeta_0)|^{3(E(\eta)+1)}}{\sup_{|\zeta|<3eR}|g(\zeta)|^{3E(\eta)} \cdot \sup_{|\zeta|<\frac{3}{2}r}|g(\zeta)|^2},$$

where R, r *and* η *are such that* $16\eta R < r$ *and* $E(\eta) = 2 + \log \frac{3e}{2\eta} > 0$.

We now state our main result.

DEFINITION 7. Let $S \in \mathcal{E}'_\omega$. S is ω-hyperbolic in $t^-(t^+)$ if there exists a fundamental solution $H^-(H^+)$ in $\mathcal{D}'_\omega$ whose support is contained in a properly convex cone in a half space $t - t_0 \leq 0 (\geq 0)$ for some $t_0 \in R$.

THEOREM 8. *Let* $S \in \mathcal{E}'_\omega$ *be invertible, i.e.,* $S * \mathcal{D}'_\omega = \mathcal{D}'_\omega$. *The following are equivalent;*

(a) *S is ω-hyperbolic in* $t^-(t^+)$.

(b) *There exists* $C_0 > 0$ *such that for* $\widehat{S}(\tau, z) = 0$,

$$Im\tau \geq -C_0(1 + |Imz| + \omega(Re(\tau, z)))$$

$$(Im\tau \leq C_0(1 + |Imz| + \omega(Re(\tau, z)))).$$

(c) *There exist positive constants* B *and* D *such that for all* $(\tau, z) \in C \times C^n$ *with* $Im\tau \leq -B(1+|Imz|+\omega(Re(\tau,z)))(Im\tau \geq B(1+|Imz|+\omega(Re(\tau,z))))$,

$$|\widehat{S}(\tau, z)| \geq \frac{1}{D} e^{-D\omega(Re(\tau,z))} e^{-D(|Im\tau|+|Imz|)}.$$

PROOF. (b) $\Rightarrow$ (c). Since S is invertible, by [1], for some positive A and C and $(\tau, z) \in C \times C^n$, if (t, x) is such that $|Re\tau - t| + |Re z - x| \leq A\omega(Re(\tau, z))$, then

$$|\widehat{S}(t,x)| \geq Ce^{-A\omega(Re(\tau,z))}.$$

Without loss of generality, we may assume $A > 1$. For any $(\tau, z) \in C \times C^n$ satisfying $Im\tau \leq -B(1 + |Im\tau| + \omega(Re(\tau, z)))$ where B will be chosen later, we apply Levin's theorem with $R = 1, r = \frac{1}{6A}, \eta = \frac{1}{97A}, \zeta_0 = 1$ and the entire function

$$g(\lambda) = \hat{S}(\tau + \lambda(t - \tau), z + \lambda(x - z)), \quad \lambda \in C,$$

where $(t, x) \in R \times R^n$ satisfies $|Re\tau - t| + |Re z - x| \leq A\omega(Re(\tau, z))$. For $|\lambda| \leq \frac{1}{4A}$ and $\zeta = (\tau + \lambda(t - \tau), z + \lambda(x - z))$,

$$\begin{aligned}
|Re\zeta| &\leq |\lambda|(|t - Re\tau| + |x - Re z|) + |Re(\tau, z)| \\
&\leq A|\lambda|\omega(Re(\tau, z)) + |Re(\tau, z)| \\
&\leq A|\lambda|M(1 + |Re(\tau, z)|) + |Re(\tau, z)| \\
&\leq \frac{M}{4} + M_2|Re(\tau, z)|,
\end{aligned}$$

for M_2 sufficiently large such that $M_2 > \frac{M}{4}$ and $\frac{\omega(\frac{M}{4})}{M_2} \leq \frac{1}{2}$,

$$|Im\zeta| \leq (1 + |Re\lambda|)|Im(\tau, z)| + A|Im\lambda|\omega(Re(\tau, z))$$

and so

$$\begin{aligned}
|\widehat{S}(\zeta)| &\leq ce^{a_0\omega(Re\zeta) + (A'+1)|Im\zeta|} \\
&\leq c'e^{(a_0M_2 + (A'+1)M_4)\omega(Re(\tau,z))} \cdot e^{(A'+1)M_3(|Im\tau| + |Im z|)},
\end{aligned}$$

for some constants c, c' and a_0. Here A' is the radius of a ball at the origin which contains the support of S. Using these estimations we have

$$\begin{aligned}
&\sup_{|\lambda| \leq 3eR}|g(\lambda)|^{3E(\eta)} \cdot \sup_{|\lambda| < \frac{3}{2}r}|g(\lambda)|^2 \\
&\leq \sup_{|\lambda| < 3e}|g(\lambda)|^{3E(\eta)+2} \\
&\leq A_1 e^{A_1\omega(Re(\tau,z))} \cdot e^{A_1(|Im\tau| + |Im z|)},
\end{aligned}$$

and

$$\begin{aligned}|\widehat{S}(\tau,z)| &= |g(0)|\\ &\geq \frac{c}{A_1}e^{-A_1\omega(Re(\tau,z))}\cdot e^{-A\omega(Re(\tau,z))}\cdot e^{-A_1(|Im\tau|+|Imz|)}\\ &\geq \frac{1}{D}e^{-D\omega(Re(\tau,z))}\cdot e^{-D(|Im\tau|+|Imz|)},\end{aligned}$$

where $D=\max\{(\frac{c}{A_1})^{-1}, A_1+A\}$, which shows (c).

It remains to show that $g(\lambda)\neq 0$ for $|\lambda|\leq\frac{1}{4A}$. For any given $\lambda=\lambda_1+i\lambda_2$ with $|\lambda|\leq\frac{1}{4A}$ let us write $T=\tau+\lambda(t-\tau), Z=z+\lambda(x-z)$. Then $ImT=(1-\lambda_1)Im\tau+\lambda_2(t-Rez)$ and $|ImZ|\leq|\lambda_2||x-Rez|+(1-\lambda_1)|Imz|$. Using $Im\tau\leq -B(1+|Imz|+\omega(Re(\tau,z)))$, $1-\lambda_1-A|\lambda_2|\leq\frac{1}{4}$ and $|t-Re\tau|+|x-Rez|\leq A\omega(Re(\tau,z))$, we can deduce

$$\begin{aligned}ImT &\leq -B((1-\lambda_1)+|ImZ|+(1-\lambda_1)\omega(Re(\tau,z)))\\ &\quad + B|\lambda_2|(|t-Re\tau|+|x-Rez|)\\ &\leq -B((1-\lambda_1)+|ImZ|+(1-\lambda_1-A|\lambda_2|)\omega(Re(\tau,z)))\\ &\quad -\frac{B}{4}(1+|ImZ|+\omega(Re(\tau,z)).\end{aligned}$$

Substituting $|Re(T,Z)|\leq\frac{M}{4}+M_2|Re(\tau,z)|$ and so $\omega(Re(T,Z))\leq\omega(\frac{M}{4})+M_2\omega(Re(\tau,z))$ into this inequality, we have

$$\begin{aligned}ImT &\leq -\frac{B}{4}\left(1+|ImZ|+\frac{1}{M_2}\left(\omega(Re(T,Z))-\omega\left(\frac{M}{4}\right)\right)\right)\\ &\leq -\frac{B}{4}\left(1+|ImZ|+\frac{1}{M_2}\omega(Re(T,Z))-\frac{\omega(\frac{M}{4})}{M_2}\right)\\ &\leq -\frac{B}{4}\left(\frac{1}{2}+|ImZ|+\frac{1}{M_2}\omega(Re(T,Z))\right)\\ &\leq -\frac{B}{8M_2}(1+|ImZ|+\omega(Re(T,Z)),\end{aligned}$$

provided that $B=\max\{8C_0M_2,1\}$. Then (b) implies $\hat{S}(T,Z)=g(\lambda)\neq 0$ for $|\lambda|\leq\frac{1}{4A}$.

(c) $\Rightarrow$ (a). For any $u \in R^n$ and $f \in \mathcal{D}_\omega$, we define H^- on $\mathcal{D}_\omega$ by

$$H^-(f) = (2\pi)^{-n} \int_{Imz=u} \int_{\Gamma_z^-} \frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)} d\tau dz,$$

where the z-integration is over $Imz = u$ and Γ_z^- is the set of τ satisfying $Im\tau = -B(1 + |Imz| + \omega(Re(\tau,z)))$. Then $H^-(f)$ is well-defined for any $u \in R^n$. Indeed, for $n = 1$, let

$$F(z) = \int_{\Gamma_z^-} \frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)} d\tau, \quad z = \xi + i\eta \in C.$$

Then since $\widehat{f}$ and $\widehat{S}$ are entire with respect to (τ, z), $F(z)$ is also analytic with respect to z. Hence by Cauchy's theorem, $\int_\gamma F(z)dz = 0$. Here $\gamma = \gamma_1 \cup \gamma_2 \cup \gamma_3 \cup \gamma_4$, where $\gamma_1, \gamma_2, \gamma_3$ and γ_4 are the lines between $(-\xi, 0)$ and $(\xi, 0)$,$(\xi, 0)$ and $(\xi, \xi + iu)$,$(\xi, \xi + iu)$ and $(-\xi, -\xi + iu)$ and $(-\xi, -\xi+iu)$ and $(-\xi, 0)$, respectively. Using the Paley-Wiener Theorem for f, the hypothesis for S and the property $(\gamma)$for $\omega(Re(\tau,z))$,

$$\begin{aligned} |F(z)| &\le \int_{\Gamma_z^-} \left| \frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)} \right| d\tau \\ &\le D'e^{(A'+D)(B+1)|u|} \int_{\Gamma_z^-} (1 + |Re(\tau,z)|)^{(-\lambda+D+(A'+D)B)b} d\tau \\ &\to 0 \text{ as } |\xi| = |Rez| \to \infty, \end{aligned}$$

if $\lambda > D + (A' + D)B$. Hence $\int_{\gamma_2} F(z)dz = \int_{\gamma_4} F(z)dz = 0$ when $\xi \to \pm\infty$. Then since $\int_{\gamma_1} F(z)dz = -\int_{\gamma_3} F(z)dz$ when $\xi \to \pm\infty$,

$$\begin{aligned} \int_{-\infty}^{+\infty} F(\xi)d\xi &= -\int_{+\infty}^{-\infty} F(\xi + iu)d\xi \\ &= \int_{-\infty}^{+\infty} F(\xi + iu)d\xi \\ &= \int_{Imz=u} \int_{\Gamma_z^-} \frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)} d\tau dz. \end{aligned}$$

Hence $H^-(f)$ is well-defined for any $u \in R$. We can clearly extend the result to n-dimension. Now, we will show that H^- is the fundamental solution of S in $\mathcal{D}'_\omega$ whose support is contained in the set $\{(t,x) : t \geq -D_1 + \frac{|x|}{B}\}$, for some positive D_1. For $f \in \mathcal{D}_\omega$ with supp$f \subset K = \{(t,x) : |(t,x)| \leq A_2$ for some $A_2\}$ and any $\lambda \in R$, by Lemma 2,

$$\begin{aligned}
|H^-(f)| &\leq \int_{Imz=u}\int_{\Gamma_z^-}\left|\frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)}\right| d\tau dz \\
&= \int_{Imz=u}\int_{\Gamma_z^-}\frac{|\widehat{f}(-\tau,-z)|}{|\widehat{S}(\tau,z)|} e^{\lambda\omega(Re(-\tau,-z))-U_K(Im(-\tau,-z)} \\
&\quad \cdot e^{-|Im(-\tau,-z)|} \cdot e^{-\lambda\omega(Re(-\tau,-z))+U_K(Im(-\tau,-z))} \\
&\quad \cdot e^{|Im(-\tau,-z)|} d\tau dz \\
&\leq \| f \|_\lambda^{(\omega)} \int_{Imz=u}\int_{\Gamma_z^-} e^{(-\lambda+D)\omega(Re(\tau,z))} \cdot e^{(A_2+1+D)|Im\tau|} \\
&\quad \cdot e^{(A_2+1+D)|Imz|} d\tau dz \\
&= \| f \|_\lambda^{(\omega)} \int_{Imz=u}\int_{\Gamma_z^-} e^{(-\lambda+D)\omega(Re(\tau,z))} \\
&\quad \cdot e^{(A_2+1+D)(B(1+|Imz|+\omega(Re(\tau,z))}\cdot e^{(A_2+1+D)|Imz|} d\tau dz \\
&= \| f \|_\lambda^{(\omega)} \int_{Imz=u}\int_{\Gamma_z^-} e^{(-\lambda+D+B(A_2+1+D))\omega(Re(\tau,z))} \\
&\quad \cdot e^{(A_2+1+D)B} \cdot e^{(A_2+1+D)(B+1)u} d\tau dz \\
&= C_u \| f \|_\lambda^{(\omega)} \int_{Imz=u}\int_{\Gamma_z^-} e^{(-\lambda+D+B(A_2+1+D))\omega(Re(\tau,z))} d\tau dz \\
&\leq C_{u,\lambda} \| f \|_\lambda^{(\omega)}, \text{if } \lambda > D + B(A_2+1+D).
\end{aligned}$$

Hence $H^- \in \mathcal{D}'_\omega$. Now for $f \in \mathcal{D}_\omega$, from the Cauchy's integral formula we have,

$$\begin{aligned}
\langle S * H^-, f\rangle &= \langle H_\xi^-, S * \check{f}(-\xi)\rangle \\
&= \int_{Imz=u}\int_{\Gamma_z^-} \frac{\widehat{S * \check{f}}(\tau,z)}{\widehat{S}(\tau,z)} d\tau dz
\end{aligned}$$

$$= \int_{Imz=u} \int_{\Gamma_z^-} \frac{\widehat{S}(\tau,z) \cdot \widehat{\check{f}}(\tau,z)}{\widehat{S}(\tau,z)} d\tau dz$$
$$= \int_{Imz=u} \int_{\Gamma_z^-} \widehat{\check{f}}(\tau,z) d\tau dz$$
$$= \int \int \widehat{\check{f}}(\tau,z) d\tau dz$$
$$= \check{f}(0,0) = \langle \delta, f \rangle.$$

Hence H^- is a fundamental solution of S. Lastly, we will prove that there exists $D_1 > 0$ such that supp$H^- \subset \{(t,x) : t \leq D_1 - \frac{|x|}{B}\}$. D_1 will be determined later. Suppose that $n = 1$. We wish to show that H^- vanishes above the line $t = D_1 - \frac{x}{B}$ and $t = D_1 + \frac{x}{B}, x > 0$. Suppose that $f \in \mathcal{D}_\omega$ and supp$f$ is contained in $t > D_1 - \frac{x}{B}$. Then for $\tau \in \Gamma_z^-, Imz < 0$,

$$\begin{aligned} |\widehat{f}(-\tau,-z)| &= |(\widehat{\check{f}})(\tau,z)| \\ &\leq \int e^{-it\tau - ixz} |\check{f}(t,x)| dt dx \\ &= \int e^{t(-B(1+|Imz|+\omega(Re(\tau,z)))+xImz} |\check{f}(t,x)| dt dx \\ &= \int e^{-tB-tB\omega(Re(\tau,z))+(tB+x)Imz} |\check{f}(t,x)| dt dx. \end{aligned}$$

Now on supp$f$, $tB + x \geq BD_1 + > 0$ and $tB \geq BD_1 - \epsilon'$ for some $\epsilon' > 0$. Since $Imz < 0$,

$$|\widehat{f}(-\tau,-z)| \leq C_f e^{-BD_1} e^{\epsilon'} \cdot e^{-BD_1(\omega(Re(\tau,z))} \cdot e^{(BD_1)Imz}.$$

On $\tau \in \Gamma_z^-$, since $Imz < 0$,

$$\left| \frac{1}{\widehat{S}(\tau,z)} \right| \leq D e^{(D+BD)\omega(Re(\tau,z))} \cdot e^{-D(B+1)Imz}.$$

Hence

$$\left|\frac{\widehat{f}(-\tau,-z)}{\widehat{S}(\tau,z)}\right| \leq C_f D e^{-BD_1} e^{\epsilon'} \cdot e^{(D+BD-BD_1)\omega(Re(\tau,z))} \cdot e^{(BD_1-D(B+1))Imz}.$$

Then by letting $Imz \to -\infty$, $H^-(f) = 0$ when D_1 is such that $BD_1 > BD + D$. This show that H^- vanishes above $t = D_1 - \frac{x}{B}, x > 0$, since $tB + x \geq BD_1 + 2x \geq BD_1 > 0$. The same method shows that H^- vanishes above $t = D_1 + \frac{x}{B}$, $x > 0$.

This completes the proof in case $n = 1$. The proof for $n > 1$ proceeds in a same manner that the lines $t = D \pm \frac{x}{B}$ are replaced by a suitable affine hyperplane.

(a) $\Rightarrow$ (b). Let H^- be a fundamental solution of S in $\mathcal{D}'_\omega$ such that suppH^- is contained in the set $\{(t,x) : t \leq D_1 - \frac{|x|}{B}\}$ for some positive D_1 and B. Let supp$S$ be contained in a compact set in R^{n+1} with $|t| \leq b_0$ for some b_0. We define $\mathcal{E}^b_\omega = \mathcal{E}_\omega(\{R^n \times t : |t| \leq b\})$. The topology of $\mathcal{E}^b_\omega$ is given by the topology in $\mathcal{E}_\omega$. For any $\epsilon > 0$, we take b with $b > 2b_0 + D_1 + \epsilon$ such that $f \in \mathcal{E}^b_\omega$ satisfies $S * f = 0$ for all t with $|t| \leq b - b_0$. Such a function f will be constructed in the last part of this implication. By Lemma 3, we can take $\varphi_\epsilon \in \mathcal{E}_\omega$ whose value is 1 on the set $\{(t,x) : |t| < b - \epsilon,\ x \in R^n\}$ and 0 on the set $\{(t,x) : |t| > b - \frac{\epsilon}{2},\ x \in R^n\}$ and let $g_\epsilon = f\varphi_\epsilon$. Then $g_\epsilon \in \mathcal{E}_\omega$ and since $g_\epsilon = f$ on $|t| < b - b_0 - \epsilon$, $S * g_\epsilon = S * f = 0$ on $|t| < b - b_0 - \epsilon$. Since supp$g_\epsilon \subset |t| < b - \frac{\epsilon}{2} < b$ and supp$S \subset |t| \leq b_0$, supp$S * g_\epsilon \subset |t| < b + b_0$. Hence we can let $S * g_\epsilon = h^1_\epsilon + h^2_\epsilon$, where supp$h^1_\epsilon \subset |t - b| < b_0 + \epsilon$ and supp$h^2_\epsilon \subset |t + b| < b_0 + \epsilon$. Since $S * g_\epsilon \in \mathcal{E}'_\omega * \mathcal{E}_\omega \subset \mathcal{E}_\omega$ by Lemma 4 and supp$h^1_\epsilon \cap$ supp$h^2_\epsilon = \emptyset$, h^1_ϵ , $h^2_\epsilon \in \mathcal{E}_\omega$. Moreover, since supp$h^1_\epsilon$ and supph^2_ϵ have compact support in t-variables, $H^- * h^2_\epsilon$ are in $\mathcal{E}_\epsilon$. Define $\bar{f} = g_\epsilon - H^- * h^2_\epsilon$. Then $\bar{f}$ has the following three properties.

(i) clearly $\bar{f} \in \mathcal{E}_\epsilon$.

(ii) $\bar{f} = f$ on $|t| < b - b_0 - \epsilon - D_1$, since supp$H^- * h^2_\epsilon \subset t \in (-\infty, -b + b_0 + \epsilon + D_1)$.

(iii)

$$\begin{aligned} S * \bar{f} &= S * g_\epsilon - S * (H^- * h^2_\epsilon) \\ &= S * g_\epsilon - (S * H^-) * h^2_\epsilon = h^1_\epsilon \\ &= 0 \ \text{ on } \ t \in (-\infty, b - b_0 - \epsilon), \end{aligned}$$

where we use the associativity of convolution that is justified by the compactness of suppH^- because of the compactness of supph^2_ϵ in t-variable. That is,

$$(i)' \ \bar{f} \in \mathcal{E}_\omega.$$
$$(ii)' \ \bar{f} = f \ \text{ on } \ |t| < b_0.$$
$$(iii)' S * \bar{f} = 0 \ \text{ on } \ t \in (-\infty, 0].$$

Now we will prove that such a function $\bar{f}$, having three properties $(i)', (ii)', (iii)'$ is unique. It suffices to show that for $h \in \mathcal{E}_\omega$, if $S * h = 0$ for $t \in (-\infty, 0]$ and $h = 0$ on $|t| < b_0$, then $h = 0$ for $t \in (-\infty, -b_0]$. Since supp$S \subset |t| \leq b_0$, supp$(S * h) \subset t < 0$. Hence $S * h = 0$ for all t. Then $h = h*\delta = h*(S*H^-) = (h*S)*H^- = 0$, where the associativity of the convolution equation is justified clearly by the support of h and H^- in t-variable. We can also show that $f \longrightarrow \bar{f}$ is continuous from the topology of $\mathcal{E}^b_\omega$ into the topology of $\mathcal{E}_\omega$ by the continuity of a convolution operator $S*$ and H^-*. Now since the embedding $\mathcal{E}_\omega \longrightarrow C^\infty_c$ is continuous, we can define a continuous linear form $f \longrightarrow \bar{f}(-2b, 0)$ on $\mathcal{E}^b_\omega(S)$ consisting of $f \in \mathcal{E}^b_\omega$ which satisfies $S * f = 0$ on $|t| < b - b_0$. Then there exists a neighborhood of 0 in $\mathcal{E}^b_\omega(S)$ on which $f \longrightarrow \bar{f}(-2b, 0)$ is bounded. Hence there exist $d > 0$ and a compact set K_1 with $K_1 \subset \{(t, x) : |x| \leq b', |t| \leq b\}$ for some $b' > 0$ such that

$$|\bar{f}(-2b, 0)| \leq d \| \ f \ \|_{\mathcal{E}^b_\omega(K_1)}.$$

We apply this result to $\bar{k}(t, x) = e^{it\tau + ixz}$ with $\widehat{S}(\tau, z) = 0$. Clearly $\bar{k} \in \mathcal{E}_\omega$ and $S * \bar{k} = 0$ for all t and x. Hence if $k(t, x) = e^{it\tau + ixz}$ with $\widehat{S}(\tau, z) = 0$ on $|t| < b$, there exists a unique $\bar{k}(t, x) = e^{it\tau + ixz}$ with $\widehat{S}(\tau, z) = 0$ for all t and x such that for some λ_0 and $\varphi_0 \in \mathcal{D}_\omega(K_1)$,

$$\begin{aligned} exp(-2bIm\tau) &= |\bar{k}(-2b, 0)| \\ &\leq d \| \ k\varphi_0 \ \|^{(\omega)}_{\lambda_0} \\ &= d \| \ e^{itRe\tau - tIm\tau + ixRez - xImz} \varphi_0 \ \|^{(\omega)}_{\lambda_0} \end{aligned}$$

$$\le de^{b|Im\tau|+b'|Imz|}\int |\widehat{e^{itRe\tau+ixRez}\varphi_0(x,t)}|$$
$$\cdot e^{\lambda_0\omega(t,x)}dtdx$$
$$\le de^{b|Im\tau|+b'|Imz|}\int |\widehat{\varphi_0}(t-Re\tau, x-Rez)$$
$$\cdot e^{\lambda_0\omega(t-Re\tau,x-Rez)}\cdot e^{\lambda_0\omega(Re(\tau,z))}dtdx$$
$$\le d'exp(b|Im\tau|+b'|Imz|+\lambda_0\omega(Re(\tau,z))).$$

Thus $Im\tau \ge -C_0(1+|Imz|+\omega(Re(\tau,z)))$. □

References

[1] S. Abdullah, *Convolution equations in the Beurling's distributions*, Acta Math. Hung **52** (1988), 7-20.

[2] G. Bjorck, *Linear partial differential operators and generalized distributions*, Ark. Math. **6** (1966), 351-407.

[3] C. C. Chou, *La Transformation de Fourier Complex et L'Equation de convolution*, Lecture Notes in Mathematics, Springer-Verlag **325** (1973).

[4] L. Ehrenpreis, *Solutions of some problems of division V*, Amer. J. M. **84** (1962), 324-348.

[5] D. H. Pahk and B. H. Kang, *Hyperbolicity in the Beurling's generalized distribution spaces*, J. Korean Math. Soc. **27** (1990), no. 1, 33-45.

Dae Hyeon Pahk
Department of Mathematics
Yonsei University
Seoul 120-749, Korea

Byung Keun Sohn
Department of Mathematics
Inje University
Kyungnam
Kimhae 621-749, Korea