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A GEOMETRIC CRITERION FOR THE WEAKER PRINCIPLE OF SPATIAL AVERAGING

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ABSTRACT. In this paper we find a geometric condition for the weaker principle of spatial averaging (PSA) for a class of polyhedral domains. Let Ω_n be a polyhedron in R^3, $n \leq 3$. If all dihedral angles of Ω_n are submultiples of π, then there exists a parallelopiped $\tilde{\Omega}_n$ generated by n linearily independent vectors $\{\mu_j\}_{j=1}^n$ in R^n containing Ω_n so that solutions of $\Delta u + \lambda u = 0$ in Ω_n with either the boundary condition $u = 0$ or $\partial u/\partial n = 0$ are expressed by linear combinations of those of $\Delta u + \lambda u = 0$ in $\tilde{\Omega}_n$ with periodic boundary condition. Moreover, if $\{\mu_j\}_{j=1}^n$ satisfies rational condition, we guarantee the weaker PSA for the domain Ω_n.

1. Introduction

In this paper we try to find a geometric condition on the domains $\Omega_n \subset R^n, n = 2, 3$, for which the principal of spatial averring (PSA) or the weaker PSA holds. The PSA was introduced by Mallet-Paret and Sell [3], [4] to prove the existence of an inertial manifold for a class of reaction diffusion equations of the form

$$u_t = \nu \Delta u + f(x, u) \tag{1.1}$$

with particular domains and boundary conditions. This property played a crucial role in their existence theory of inertial manifold. For their result, they proved that PSA holds for arbitrary rectangular and a cubic domains of the equation (1.1) and Kwean [2] found a weaker form of PSA so that he extended their result into several different types of domains.

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But it is not known for which domains and boundary conditions PSA or the weaker PSA holds.

Here we introduce some theories of discrete group of isometries, which are applied for a class of polyhedral domains to decide whether PSA or the weaker PSA holds. For this purpose, we briefly introduce some terminology and preliminary results about the discrete group of isometries of E^n and then from these results, one obtains some sufficient conditions to guarantee the weaker PSA.

2. Definitions and preliminary results

In this section, we study basic properties of discrete groups of isometries.

Let E^n be Euclidean n-space and $I(E^n)$ the set of isometries of E^n. Then with the multiplication defined by composition, $I(E^n)$ forms a group, called the group of isometries. Moreover, if $I(E^n)$ is given with the subspace topology inherited from the space $C(E^n, E^n)$ of continuous self-map of E^n with compact-open topology, it becomes a topological group. By a discrete group, we mean a topological group with discrete topology.

Now throughout the rest of this paper, we consider a finite-sides convex polyhedron P in E^n of finite volume. Let S be a side of P. The reflection of E^n in the side S of P is the reflection of E^n in the hyperplane $\langle S \rangle$ spanned by S. If G is the group generated by reflections of E^n in each sides S of P, G is also a topological group as a subgroup of $I(E^n)$. In particular, we are interested in the relationship of convex polyhedron with a discrete group and a discrete group.

DEFINITION. A convex fundamental polyhedron for a discrete group G of isometries of E^n is a convex polyhedron P in E^n such that

(1) the interior, P°, of P is a connected set in E^n,
(2) $\{g(P^\circ) : g \in G\}$ is a locally finite family of mutually disjoint subsets of E^n,
(3) $E^n = \cup\{g(P) : g \in G\}$.

In addition, P is an exact, convex, fundamental polyhedron for a discrete group G if for each side S of P, there is an element $g_S \in G$ such that

$$S = P \cap g_S(P).$$

Moreover, g_S is the reflection of E^n in the hyperplane $\langle S \rangle$ for each side S of P if and only if the group G is a discrete reflection group with respect to the polyhedron P.

For a given polyhedron P and the group G generated by reflection of E^n in the sides of P, our primary objective is to find a geometric condition on the polyhedron P so that G is a discrete reflection group and P is an exact, convex, fundamental polyhedron of a discrete reflection group G. For this purpose, we introduce some terminologies; let S and T be sides of polyhedron P. Sides S and T are said to be adjacent if $S \cap T$ is a side of both S and T. When S and T are adjacent sides of P, we define a dihedral angle $\theta(S,T)$ of P to be the angle formed by S and T. Moreover, an angle α is a submultiple of an angle β if and only if there is a positive integer n such that $\alpha = \frac{\beta}{n}$.

With above definitions, one has the following two important results and the proofs can be found in J. G. Ratcliffe [5]. However, for the convenience of the reader, we present the basic ideas for the proofs.

Proposition 2.1. *Let P be a polyhedron in E^n and let G be a group generated by reflections of E^n in sides of P. Then G is a discrete reflection group with respect to the polyhedron P if and only if all the dihedral angles of P are submultiples of π.*

Proof. Assume that G is a discrete reflection group of P. Let S and T be adjacent sides of P. Then $\{S,T\}$ is a cycle of sides of P. Then there exists a positive integer k such that

$$2\,\theta(S,T) = \frac{2\pi}{k}$$

and hence $\theta(S,T)$ is a submultiple of π.

Conversely, assume that all dihedral angles are submultiples of π. The proof is given by the induction on n. The conclusion is true for $n = 1$, so we assume that it is true in dimension $n-1$. The idea of the proof is to construct a topological space $\tilde{X}$ for which the conclusion is true and then to show that $\tilde{X}$ is homeomorphic to E^n by the covering space argument.

Let $G \times P$ be the cartesian product of G and P. We topologize $G \times P$ by giving G the discrete topology and $G \times P$ the product topology. Then $G \times P$ is the topological sum of the subspaces

$$\{\{g\} \times P : g \in G\}.$$

Moreover, the mapping $(g, x) \to gx$ is a homeomorphism of $\{g\} \times P$ onto gP for each $g \in G$.

Let $\mathcal{S}$ be the set of sides P and for each $S \in \mathcal{S}$ and g_S be the reflection of E^n in the side S of P. Let $\Phi = \{g_S : S \in \mathcal{S}\}$. Now we define a relation on $G \times P$ as follows: two points (g, x) and (h, y) are said to be paired by Φ, written $(g, x) \backsim (h, y)$, if and only is $g^{-1}h$ is in Φ and $gx = hy$. Then this relation becomes an equivalence relation on $G \times P$. Let $[g, x]$ be the equivalence class of (g, x) and $\tilde{X}$ the quotient space of $G \times P$ of equivalence classes. Then we have the following properties:

(1) each $g[P] = \{[g, x] : x \in P\}$ is connected
(2) $\{g(P^\circ) : g \in G\}$ is a locally finite family of mutually disjoint subsets of $\tilde{X}$,
(3) $\tilde{X} = \cup\{g[P] : g \in G\}$.

Now we define a function $\kappa : \tilde{X} \to E^n$ by $\kappa[g, x] = gx$. Then this function κ becomes homeomorphism and hence we have a family $\mathcal{P} = \{g(P) : g \in G\}$ satisfying the same properties (1), (2), and (3) in E^n. Then G is discrete and P is an exact, convex, fundamental polyhedron for G. Therefore, G becomes a discrete reflection group for P. □

DEFINITION. An n-dimensional crystallographic group is a discrete group G of isometries of E^n such that E^n/G is compact.

Then, by the result of Proposition 2.1, the discrete reflection group G in Proposition 2.1 is a n-dimensional crystallographic group. We also obtain the following result.

PROPOSITION 2.2. *Let G be a discrete group of isometries of E^n. Then G is crystallographic if and only if the subgroup N of translations of G is of finite index and has rank n.*

PROOF. Suppose that G is a crystallographic. Then G has an abelian discrete subgroup H of finite index containing N. Since E^n/G is compact if and only if E^n/H is compact, H is also crytallographic. Moreover, from the facts that E^n/H is compact and that H is an abelian discrete group, H must be a lattice subgroup of $I(E^n)$ generated by n linearly independent translations. Hence $H = N$, and N is of finite index in G and has rank n.

Conversely, suppose that the subgroup N of translations of G is of finite index and has rank n. We choose a basis $v_1, \cdots, v_n$ of R^n such that N is the group generated by the translation of E^n by $v_1, \cdots, v_n$. Clearly, the parallelopiped P spanned by $v_1, \cdots, v_n$ is a convex fundamental polyhedron for N. Since P is compact, E^n/N is compact. Therefore, E^n/G is compact. □

From the conclusions of Proposition 2.1 and 2.2, we obtain the following theorem for $n = 3$.

THEOREM 2.3. *Let P be a polyhedron whose all dihedral angles are submultiples of π. Let G be a group generated by reflections of E^3 in sides of P. Then there exists a parallelopiped $\tilde{P}$ satisfying the following conditions:*

(1) $\tilde{P} = \{x \in R^3 : x = \sum_{j=1}^{3} x_j\mu_j, 0 \leq x_j \leq 1\}$ *where μ_j's are linearly independent vectors in R^3,*

(2) $P \subset \tilde{P}$,

(3) *if w_1, w_2 are boundary points of $\tilde{P}$ such that $w_2 = w_1 + \mu_j$ for some $j = 1, 2, 3$, then there exist $x \in P$ and $g_1, g_2 \in G$ such that*

$$w_1 = g_1(x), \quad w_2 = g_2(x).$$

PROOF. By the results of Proposition 2.1 and 2.2, G is a 3-dimensional crystallographic group and therefore G has a subgroup N containing all translations in G. Moreover, since N has rank 3, there exists a basis w_1, w_2, w_3 in R^3 which generates the subgroup N. For simplicity, we may assume that the polyhedron P lies on the first quadrant and it has three vertices v_0, v_1, v_2 in xy-plane. In particular, we may assume that v_0 is the origin, v_1 is a point on x-axis and the origin is connected by only three vertices of P which are denoted by v_1, v_2 and v_3. We also let S_1, S_2, S_3 be the sides of P containing triple-vertices $(0, v_2, v_3), (0, v_1, v_2), (0, v_1, v_3)$, respectively, and let P_j be the plane in R^3 containing each side S_j of P.

We want to show that there exist translations ϕ_j in N and vectors μ_j such that

(1) $\phi_j(x) = x + \mu_j$ for $x \in R^3$ and $j = 1, 2, 3$,

(2) the vectors $\overrightarrow{0\mu_j}$'s are linearly independent in R^3 such that $\overrightarrow{0\mu_j}$ is parallel to $\overrightarrow{0v_j}$.

For this purpose, we let T_{nj} be a translation in N such that for each integer $n \in Z$ and $j = 1, 2, 3$,

$$T_{nj}x = x + nw_j, \quad \text{for all} \quad x \in R^3.$$

First, we consider the collection of translations of the form $\{T_{n1} : n \in Z\}$. If $\overrightarrow{0w_1} // \overrightarrow{0v_1}$, we take w_1 for μ_1. Otherwise, for each n, each translation T_{n1} maps the plane P_1 to another plane denoted by P_{n1}. Then each P_{n1} meets at one point z_{n1} with the planes P_2, P_3 and then each z_{n1} belongs to a G-orbit, $G_{v_j} = \{g(v_j) : g \in G\}$, where v_j is a vertex of P. Moreover all dihedral angles at z_{n1}, $\theta(P_{n1}, P_2), \theta(P_{n1}, P_3)$, $\theta(P_2, P_3)$, are the same as those at the origin, $\theta(P_1, P_2), \theta(P_1, P_3)$, $\theta(P_2, P_3)$. Therefore if $z_{n1} \in G_{v_j}$, either v_j is the origin or v_j is a vertex of P satisfying above property. Since P is a polyhedron, there are only finite number of such vertices of P. Hence, if $z_{11} \notin G_0$, there exists a positive integer n_1 such that $z_{n_1 1} \in G_0$. Let P' be a copy of P such that it contains $z_{n_1 1}$ and it has three edges e_j connected with $z_{n_1 1}$ which are parallel to and have the same direction with $\overrightarrow{0v_j}$. Also let r_j be a vertex of P' which is the other endpoint of e_j.

On the other hand, since $R^3 = \cup\{g(P) : g \in G\}$, there exists an element $\phi : P \to P'$ in G such that $\phi(0) = z_{n_1 1}$. Suppose that $\phi(v_j) = r_j$ for each j. We claim that ϕ is a translation in G and hence in N; since ϕ is a composite of finitely many reflections, ϕ is of the form

$$\phi(x) = Ax + b \quad \text{for} \quad x \in R^3$$

where A is 3×3 invertible matrix and b is a vector in R^3. We note that any reflections leave all distance and angles invariant but they are orientation reversing. Then since $\phi(v_j) = r_j$ for each j, $r_j = v_j + z_{n_1 1}$ and hence $b = z_{n_1 1}$ and $A = I$. Therefore, ϕ must be a translation. Moreover, if $y_1 \in P_1$, then $y_1 + z_{n_1 1} \in P_{n_1 1}$. Now there exists a point $x \in \partial \bar{P}$ and an element $g \in G$ such that $y_1 = g(x)$ and hence implies $y_1 + z_{n_1 1} = \phi \circ g(x)$ for $\phi \circ g \in G$. Therefore, the planes P_1 and $P_{n_1 1}$ satisfy the 3^{rd} requirement of Theorem 2.3. Now for ϕ_1, we take $\mu_1 = z_{n_1 1}$.

If $\phi(v_j) \neq r_j$ for some j, then ϕ just changes left and right of P and hence P' has different left and right at the point $z_{n_1 1}$ with those of P at the origin. But there are only finitely many such cases. Hence, we obtain another copy P'' of P which has the same left and right at some point $z_{m1} \in G_0$ as of P. Thus we have a desired translation $\phi : P \to P''$.

Similarly, we can do the same procedure with w_2, w_3 so that we obtain translations of the form $\phi_j(x) = x + \mu_j, x \in R^3$ with $\overrightarrow{0\mu_j} // \overrightarrow{0v_j}$. Now we define a polyhedron $\tilde{P}$ in R^3 as follows:

$$\tilde{P} = \left\{ x \in R^3 : x = \sum_{j=1}^{3} x_j \mu_j, \quad 0 \leq x_j \leq 1 \right\}.$$

Then the polyhedron $\tilde{P}$ satisfies all requirements of Theorem 2.3. □

REMARK. We call the requirements (1), (2), (3) of Theorem 2.3 a P-property and $\tilde{P}$ a parallelopiped generated by μ_1, μ_2, μ_3 with P-property for P.

3. The weaker PSA

Now we introduce the definition of the weaker PSA. Let $H = L^2(\Omega)$ where $\Omega \subset R^n$ is a bounded domain. For any $\lambda > 0$ let P_λ denote the canonical orthogonal projection onto the finite dimensional subspace

$$\mathcal{P}_\lambda = Span\{e_m : \lambda_m \leq \lambda\}$$

of H where $\{e_j : j = 1, 2, \cdots\}$ be complete orthonormal set of eigenfunctions e_j corresponded to eigenvalues λ_j of $-\Delta$ with a given choice of boundary conditions and let $Q_\lambda = I - P_\lambda$.

For any $v \in L^\infty$ we let B_v denote the operator on L^2 defined by

$$(B_v u)(x) = v(x)u(x), \qquad u \in L^2$$

and let $\tilde{v}$ denote the mean value

$$\tilde{v} = (\mathrm{vol}\Omega)^{-1} \int_\Omega v(x)dx.$$

Then the weaker PSA gives a comparison between the vector fields induced by B_v and $\tilde{v}I$ on the finite dimensional range of $(P_{\lambda+\kappa} - P_{\lambda-\kappa})$. Now we define a weaker PSA as follows.

Definition. For a given (bounded Lipschitz) domain $\Omega \subset R^n, n \leq 3$, and choice of boundary conditions for the Laplacian, we say the weaker principle of spacial averaging holds if there exists a quantity $\xi > 0$ such that for every $\epsilon > 0$, $\kappa > 0$ and any bounded subset $\mathcal{B} \subset H^2(\Omega)$, there exists arbitrarily large $\lambda = \lambda(\mathcal{B}) > \kappa$, such that

$$||(P_{\lambda+\kappa} - P_{\lambda-\kappa})(B_v - \tilde{v}I)(P_{\lambda+\kappa} - P_{\lambda-\kappa})||_{op} \leq \epsilon \tag{3.1}$$

holds for any $v \in \mathcal{B}$; and such that

$$\lambda_{m+1} - \lambda_m \geq \xi \tag{3.2}$$

where m satisfies $\lambda_m \leq \lambda < \lambda_{m+1}$.

Since the weaker PSA and PSA heavily depend on the eigenvalues and the eigenfunctions of $-\Delta$, it is natural to consider the eigenvalue problem

$$\Delta u + \lambda u = 0 \quad \text{in } \Omega \tag{3.3}$$

and we consider one of the following boundary conditions for the equation (3.3):

$$\begin{cases} \text{Dirichlet :} & u = 0 \quad \text{on } \partial\Omega \\ \text{Neumann :} & \dfrac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{3.4}$$

We also consider periodic boundary conditions when the domain is a cartesian product of intervals.

First of all, we consider a special case. Let Ω be given by

$$\Omega = \left\{ x \in R^3 : \; x = \sum_{j=1}^{3} x_j \mu_j, 0 \le x_j \le 1 \right\} \tag{3.5}$$

where μ_j's are linearly independent vectors in R^3 and we consider Ω-periodic boundary conditions for the equation (3.3), that is, if $\partial\Omega = \cup_{j=1}^{6} \Gamma_j$ such that $\Gamma_{j+3} = \Gamma_j + \mu_j$, then

$$u(x) = u(x + \mu_j), \qquad x \in \Gamma_j \quad \text{for } j = 1, 2, 3. \tag{3.6}$$

For $\mu_j = (\mu_{j1}, \mu_{j2}, \mu_{j3}) \in R^3$, let A be a column matrix of μ_1, μ_2, μ_3.

By using the symmetric property (3.6) and computing the calculation, one obtains the next lemma.

LEMMA 3.1.. *The eigenvalues and the eigenfunctions of* $-\Delta$ *for* (3.5), (3.6) *are of the form: for each* $k = (k_1, k_2, k_3) \in Z^3$,

$$\lambda_k = \frac{4\pi^2}{|A|^2} |k_1\sigma_1 + k_2\sigma_2 + k_3\sigma_3|^2 \tag{3.7}$$

$$f_k(x, y, z) = \exp\left(\frac{2\pi i}{|A|}\right) (\alpha x + \beta y + \gamma z) \tag{3.8}$$

where $|\cdot|$ *is the usual norm in* R^3 *and* σ_j*'s and* (α, β, γ) *are given by*

$$\begin{cases} \sigma_1 = \mu_2 \times \mu_3, \quad \sigma_2 = \mu_3 \times \mu_1, \quad \sigma_3 = \mu_1 \times \mu_2, \\ |A| = det \begin{pmatrix} \mu_{11} & \mu_{12} & \mu_{13} \\ \mu_{21} & \mu_{22} & \mu_{23} \\ \mu_{31} & \mu_{32} & \mu_{33} \end{pmatrix}, \quad \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = A^{-1} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}. \end{cases} \tag{3.9}$$

The following lemmas play important roles for proving the weaker PSA. First, we define a new inner product and a norm in R^3. For $x, y \in R^3$, we define

$$\langle x, y \rangle = \left(\sum_{j=1}^{3} x_j \delta_j \right) \cdot \left(\sum_{i=1}^{3} y_i \delta_i \right), \quad [x]^2 = \langle x, x \rangle$$

where " $\cdot$ " is the usual inner product in R^3 and $\delta_j = \frac{\sigma_j}{|\sigma_3|}$ for each $j = 1, 2, 3$. Here the choice of σ_3 can be replaced by σ_1 or σ_2. Then for each $k \in Z^3$, the eigenvalues in (3.7) can be expressed by

$$\lambda_k = \frac{4\pi^2}{|A|^2} |\sigma_3|^2 [k]^2.$$

Although the next lemma is modified from one of Mallet-Paret and Sell [4], it is worth proving it because we are dealing with a different type of lattice points in R^3.

LEMMA 3.2. *Suppose that $(\sigma_i \cdot \sigma_j)/(\sigma_{i'} \cdot \sigma_{j'})$ is rational number whenever $\sigma_{i'} \cdot \sigma_{j'} \neq 0$ for $i, j, i', j' = 1, 2, 3$. For each $k \in Z^3$, let $[k]^2 = \langle k, k \rangle$. Then there exists a quantity $\xi > 0$ such that for any given $\kappa > 0, d > 1$, there exists arbitrary large $\lambda > 0$ such that the following statements hold:*

(1) *if $[k]^2, [l]^2 \in (\lambda - \kappa, \lambda + \kappa]$, one has either $k = l$ or $[k - l] \geq d$; and*
(2) *$[k]^2 \notin (\lambda - \frac{\xi}{2}, \lambda + \frac{\xi}{2})$ for $k \in Z^3$.*

PROOF. Since $(\sigma_i \cdot \sigma_j)/(\sigma_{i'} \cdot \sigma_{j'})$ is rational number, each $\delta_i \cdot \delta_j$ is also rational number and hence let $\delta_i \cdot \delta_j \equiv \frac{q_{ij}}{p_{ij}}$ where p_{ij} and q_{ij} are relative prime integers. Let $\alpha = L.C.M.\{p_{ij}\}$ be fixed where L.C.M. means least common multiple. Then for any $k \in Z^3$, there exist integers n and r such that

$$[k]^2 = n + \frac{r}{\alpha}, \qquad 0 \leq r < \alpha.$$

Therefore, with $\xi = \frac{1}{2\alpha}$ we see that there are arbitrarily large λ such that $[k]^2 \notin (\lambda - \frac{\xi}{2}, \lambda + \frac{\xi}{2})$. For the rest of the proof we will consider only such λ. Let λ be fixed and let N_0^λ be the annular region

$$N_0^\lambda \equiv \{x \in R^3 : \lambda - \kappa < [x]^2 \leq \lambda + \kappa\}.$$

Suppose that $k,\ l \in N_0^\lambda \cap Z^3$ and $0 < [k - l] < d$. Then for $j = l - k$,

$$[l]^2 = [j + k]^2 = [j]^2 + 2\langle k, j \rangle + [k]^2,$$

where $\langle \cdot, \cdot \rangle$ is defined. As a result, one obtains

$$\begin{aligned}|\langle k, j\rangle| &\leq \frac{1}{2}|\ [l]^2 - [k]^2 - [j]^2\ | \\ &\leq \frac{1}{2}|\ [l]^2 - [k]^2\ | + \frac{1}{2}[j]^2 \\ &< \kappa + \frac{d^2}{2}.\end{aligned}$$

For each j with $0 < [j] < d$, let $C_j = \{x \in R^3 : |\langle x, j\rangle| < \kappa + \frac{d^2}{2}\}$ and let $C = \bigcup_{0<[j]<d} C_j$. Then $k \in C_j$ for some j. If (1) fails for this choice of λ, then

$$C \cap N_0^\lambda \cap Z^3 \neq \emptyset.$$

Consider $k \in C \cap N_0^\lambda \cap Z^3$. Then one has

$$|\langle k, j\rangle| < \kappa + \frac{d^2}{2}$$

and

$$\begin{aligned}\gamma &\equiv \langle k, j\rangle \\ &= k_1\delta_1 \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3) + k_2\delta_2 \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3) \\ &\quad + k_3\delta_3 \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3)\end{aligned}$$

for some j and some $\gamma = \frac{n}{\alpha}$ where $0 < [j] < d$ and n is an integer such that $|\frac{n}{\alpha}| < \kappa + \frac{d^2}{2}$. Since $\gamma = \frac{n}{\alpha}$ for some integer n, there is only a finite number of γ satisfying $|\gamma| = |\frac{n}{\alpha}| < \kappa + \frac{d^2}{2}$. On the other hand since $j \neq 0$, one has

$$\delta_i \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3) \neq 0$$

for some $i = 1, 2, 3$. Otherwise, $j_1\delta_1 + j_1\delta_2 + j_3\delta_3$ is either orthogonal to all δ_i or equal to 0. Since $\{\delta_1, \delta_2, \delta_3\}$ generates R^3, both cases are reduced to the case $j_1\delta_1 + j_1\delta_2 + j_3\delta_3 = 0$ which contradicts that $j \neq 0$. So we may assume that

$$\delta_3 \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3) \neq 0.$$

Then by solving $\langle k, j\rangle = \gamma$ for k_3, one finds

$$k_3 = \frac{\gamma}{\rho_3} - \frac{\rho_1}{\rho_3}k_1 - \frac{\rho_2}{\rho_3}k_2$$

where $\rho_i = \delta_i \cdot (j_1\delta_1 + j_2\delta_2 + j_3\delta_3)$ for each $i = 1, 2, 3$. Hence by substituting k_3, one has

$$\begin{aligned}[k]^2 =& k_1^2|\delta_1 - \frac{\rho_1}{\rho_3}\delta_3|^2 + k_2^2|\delta_2 - \frac{\rho_2}{\rho_3}\delta_3|^2 + 2k_1k_2(\delta_1 - \frac{\rho_1}{\rho_3}\delta_3)\cdot(\delta_2 - \frac{\rho_2}{\rho_3}\delta_3) \\ &+ 4k_1(-\frac{\gamma\rho_1}{\rho_3^2}|\delta_3|^2 + \frac{\gamma}{\rho_3}\delta_1\cdot\delta_3) + 4k_2(-\frac{\gamma\rho_2}{\rho_3^2}|\delta_3|^2 + \frac{\gamma}{\rho_3}\delta_2\cdot\delta_3) \\ &+ \frac{\gamma^2}{\rho_3^2}|\delta_3|^2.\end{aligned}$$

Then we define a function $T_{j,\gamma}$ of quadratic forms on Z^2 with rational coefficients by

$$T_{j,\gamma}(l_1, l_2) = a_j l_1^2 + b_j l_1 l_2 + c_j l_2^2 + s_{j,\gamma} l_1 + t_{j,\gamma} l_2 + r_{j,\gamma} \tag{3.10}$$

where

$$\begin{cases} a_j = |\delta_1 - \frac{\rho_1}{\rho_3}\delta_3|^2, \ b_j = 2(\delta_1 - \frac{\rho_1}{\rho_3}\delta_3)\cdot(\delta_2 - \frac{\rho_2}{\rho_3}\delta_3), \\ c_j = |\delta_2 - \frac{\rho_2}{\rho_3}\delta_3|^2, \ s_{j,\gamma} = 4(\frac{\gamma}{\rho_3}\delta_1\cdot\delta_3 - \frac{\gamma\rho_1}{\rho_3^2}|\delta_3|^2), \\ t_{j,\gamma} = 4(\frac{\gamma}{\rho_3}\delta_2\cdot\delta_3 - \frac{\gamma\rho_2}{\rho_3^2}|\delta_3|^2), \ r_{j,\gamma} = \frac{\gamma^2}{\rho_3^2}|\delta_3|^2 \end{cases} \tag{3.11}$$

Then the discriminant of $T_{j,\gamma}$ in (3.10) is

$$b_j^2 - 4a_jc_j = 4\left[\left(\delta_1 - \frac{\rho_1}{\rho_3}\delta_3\right)\cdot\left(\delta_2 - \frac{\rho_2}{\rho_3}\delta_3\right)\right]^2 - 4\left|\delta_1 - \frac{\rho_1}{\rho_3}\delta_3\right|^2\left|\delta_2 - \frac{\rho_2}{\rho_3}\delta_3\right|^2.$$

Let $\tau_1 = \delta_1 - \frac{\rho_1}{\rho_3}\delta_3, \tau_2 = \delta_2 - \frac{\rho_2}{\rho_3}\delta_3$. Then

$$b_j^2 - 4a_jc_j = -4|\tau_1|^2|\tau_2|^2(1 - \cos^2\theta_{\tau_1\tau_2}) \leq 0$$

where the angle $\theta_{\tau_1\tau_2}$ is one made by τ_1 and τ_2. Moreover, since the vectors $\{\delta_j\}$ are linearly independent, the equality cannot hold. Hence the discriminant of $T_{j,\gamma}$ is negative. One then has

$$T_{j,\gamma}(k_1,k_2) \quad \in \quad (\lambda-\kappa,\lambda+\kappa] \tag{3.12}$$

since $k \in N_0^\lambda$.

Now let $\mathcal{T} = \{T_{j,\gamma} : 0 < [j] < d, |\gamma| = |\frac{n}{\alpha}| < \kappa + \frac{d^2}{2}$, for some $n \in \mathbb{Z}\}$. Then since the indices j and γ range over finite sets, $\mathcal{T}$ is a finite collection of functions $T_{j,\gamma}$ defined by (3.10), (3.11). Then by Spectral Gap Theorem proved by Mallet-Paret and Sell [4] (also see [6]), given any $h > 0$ there exists arbitrary large m such that

$$T_{j,\gamma} \notin [m,m+h], \quad \text{for} \quad T_{j,\gamma} \in \mathcal{T} \quad \text{and} \quad (l_1,l_2) \in Z^2.$$

Therefore, with $h = 2+2\kappa$, there exists a m such that for any $T_{j,\gamma} \in \mathcal{T}$ and $l \in Z^2$

$$T_{j,\gamma}(l_1,l_2) \notin [m,m+h], \quad (\lambda-\kappa,\lambda+\kappa] \subset [m,m+h]$$

for some λ satisfying the second assertion (2). Therefore (3.12) is impossible for this λ. As m can be chosen arbitrarily large, the proof is now complete. □

REMARK. We call the condition on $\{\sigma_j\}$ rational conditions. When the domain is a cubic domain, then this condition is reduced to the rational condition of Mallet-Paret and Sell [4].

Another lemma we introduce is given as follows.

LEMMA 3.3. *Let $\Omega \subset R^3$ be given in* (3.5). *Fix boundary conditions for $-\Delta$ given in* (3.6) *and let $\mathcal{B}$ be a bounded subset of $H^2(\Omega)$. Then for any $\epsilon > 0$ and $\kappa > 1$, there exists arbitrarily large $\lambda = \lambda(\mathcal{B}) > \kappa$ such that*

$$\left| \int_\Omega (v-\tilde{v})\rho^2 dx \right| \leq \epsilon \tag{3.13}$$

for any $v \in \mathcal{B}$ and $\rho \in$ Range $(P_{\lambda+\kappa} - P_{\lambda-\kappa}) \subset L^2(\Omega)$ with $||\rho|| = 1$.

PROOF. We note that the product of any two eigenfunctions of the form in (3.8) is also eigenfunction, i.e.,

$$f_k \bar{f}_l = f_{k-l} \tag{3.14}$$

where $\bar{f}$ means the complex conjugate of f. With the property (3.14), the result follows from the property (1) of Lemma 3.2 and the facts that the set of eigenfunctions of Laplace operator forms complete orthogonal basis for L^2 and that any bounded set of H^2 is compact subset of L^2 for $n \leq 3$. For more detailed proof, we mention Kwean [2]. □

Combining the results of Lemma 3.1-3.3, one obtains the following.

LEMMA 3.4. *The weaker PSA hold for the domain and boundary conditions in* (3.5), (3.6).

PROOF. We fix a quantity $\xi > 0$ satisfying the property (2) of Lemma 3.2. Let $\epsilon > 0, \kappa > 0$, and a bounded subset $\mathcal{B} \subset H^2(\Omega)$ be given. Then we have arbitrarily large $\lambda > \kappa$ satisfying the property (1) of Lemma 3.2 and the inequality (3.13) in Lemma 3.3. Therefore the inequalities (3.1) and (3.2) can be obtained by the choices of $\xi > 0$ and λ. □

Finally, one can formulate a condition on the domain for the weaker PSA as follows.

THEOREM 3.5. *Let P be a polyhedron in $R^n, n = 2, 3$, whose all dihedral angles are submultiplies of π. Let $\tilde{P}$ be a parallelopiped generated by μ_1, μ_2, μ_3 with P-property for P. Then for $n = 2$, the weaker PSA holds for P with boundary conditions in* (3.4). *For $n = 3$, one has the same conclusion provided $\{\sigma_j\}$ generated by $\{\mu_j\}$ satisfies rational conditions.*

When $n = 2$, P is a polygon in R^2 and there are only four cases satisfying the assumption which are

$$\begin{cases} (\theta_1, \theta_2, \theta_3) = \left(\frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{6}\right), \quad \left(\frac{\pi}{2}, \frac{\pi}{4}, \frac{\pi}{4}\right), \quad \left(\frac{\pi}{3}, \frac{\pi}{3}, \frac{\pi}{3}\right), \\ (\theta_1, \theta_2, \theta_3, \theta_4) = \left(\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}\right). \end{cases} \tag{3.15}$$

For $n = 3$, we consider the next lemma for the Dirichlet boundary conditions.

LEMMA 3.6. *Let P and $\tilde{P}$ be polyhedrons given in* Theorem 3.5 *and let R_j be a reflection of R^3 for each side S_j of P. There exists a canonical isomorphism between $L^2(P)$ and a subspace $\mathcal{V}$ of $L^2(\tilde{P})$,*

$$\mathcal{V} = \{\tilde{f} \in L^2(\tilde{P}) : R_j\tilde{f} = -\tilde{f} \text{ for all } R_j\}$$

obtained by $\tilde{f} \to \tilde{f}|_P$ for $\tilde{f} \in \mathcal{V}$.

PROOF. Let G be a group generated by reflections of E^n in sides of P. Since P is a parallelopiped satisfying P-property for P, $\tilde{P}$ is the union of a finite number of copies of P by elements of G, i.e.,

$$\tilde{P} = \bigcup_{j=1}^{m} g_j(P)$$

for some $g_j \in G$ and some integer m. Since $g_j \in G$, let g_j be the composition of k_j number of reflections of E^n in sides of P. Then if $\tilde{f}$ is in $\mathcal{V}$, one has, for each $x \in P$ and each j,

$$\tilde{f}(x) = \begin{cases} -\tilde{f}(g_j(x)) & \text{if} \quad k_j = \text{ odd} \\ \tilde{f}(g_j(x)) & \text{if} \quad k_j = \text{ even.} \end{cases} \tag{3.16}$$

Hence the value and the L^2–norm of $\tilde{f} \in \mathcal{V}$ on $\tilde{P}$ determine those of $\tilde{f}$ on any $g_j(P)$, say P. Conversely, if f is in $L^2(P)$, then one has a unique extension $\tilde{f}$ of f up to $\tilde{P}$ by the rule (3.16). Then $\tilde{f}$ is in $\mathcal{V}$ and f decides the value and L^2-norm of $\tilde{f}$ on $\tilde{P}$. In this sense, a map from $\mathcal{V}$ to $L^2(P)$ obtained by $\tilde{f} \to \tilde{f}|_P$ becomes an isomorphism. □

In particular, the eigenfunction of $-\Delta$ on P can be obtained by solving the equation (3.3) on $\mathcal{V}$ and the restriction to P will automatically satisfy the Dirichlet boundary conditions. Therefore the eigenfunctions of $-\Delta$ on P are given by linear combination of the form

$$\tilde{f}(x, y, z) = \exp\left(\frac{2\pi i}{|A|}\right)(\alpha x + \beta y + \gamma z) \tag{3.17}$$

where α, β, γ and $|A|$ are of the form in (3.9).

For the Neumann boundary conditions, for given solution f on P of (3.3), we can lift f to a function $\tilde{f}$ on $\tilde{P}$ satisfying

$$R_j\tilde{f} = \tilde{f}, \quad x \in P, \quad \text{for all } \ R_j;$$

$\tilde{f}|_P = f$. $\tilde{f}$ will still be an eigenfunction on P and hence a linear combination of (3.17) with the same values of (α, β, γ). Hence we have the same result as one of Lemma 3.6 with

$$\mathcal{V}' = \{\tilde{f} \in L^2(\tilde{P}) : R_j\tilde{f} = \tilde{f} \text{ for all } R_j\}.$$

PROOF OF THEOREM 3.5. For $n = 2$, P is one of triangles and rectangle in (3.15) and these cases are already known by Mallet-Paret and Sell [4] and Kwean [2]. For $n = 3$, the result comes from the Lemma 3.4 because any eigenfunction of $-\Delta$ on P is given by a linear combination of the functions in (3.17). □

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