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WHITE NOISE HYPERFUNCTIONS

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ABSTRACT. We construct the Gelfand triple based on the space $\mathcal{G}$, introduced by Sato and di Silva, of analytic and exponentially decreasing functions. This space denoted by $(\mathcal{G})$ of white noise test functionals are defined by the operator $\cosh\sqrt{A}$, $A = -\left(\frac{d}{dx}\right)^2 + x^2 + 1$. We also note that many properties like generalizations of the Paley–Wiener theorem and the Bochner–Schwartz theorem hold in this space as in the space of Hida distributions.

1. Introduction

Let $\mathcal{S}(\mathbb{R})$ be the Schwartz space of real valued rapidly decreasing functions. Then its dual space $\mathcal{S}'(\mathbb{R})$ consists of the tempered distributions and we have the Gelfand triple

$$\mathcal{S}(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'(\mathbb{R}).$$

Let μ be the standard Gaussian measure on $\mathcal{S}'(\mathbb{R})$, i.e., its characteristic function is given by

$$\int_{\mathcal{S}'(\mathbb{R})} e^{i\langle x,\xi\rangle} d\mu(x) = e^{-|\xi|_0^2/2}, \quad \xi \in \mathcal{S}(\mathbb{R}),$$

where $\langle\cdot,\cdot\rangle$ is the pairing of $\mathcal{S}(\mathbb{R})$ and $\mathcal{S}'(\mathbb{R})$, and $|\cdot|_0$ is the norm of $L^2(\mathbb{R})$. We call the space $(\mathcal{S}'(\mathbb{R}), d\mu)$ the white noise space of Hida.

On the other hand, we introduce the real versions of the space $\mathcal{G}$ and $\mathcal{F}$ of test functions for the Fourier ultrahyperfunctions and the Fourier hyperfunctions respectively in [4,7], which are both invariant under Fourier transform as follows.

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DEFINITION 1.1. We denote by $\mathcal{G}$ or $\mathcal{G}(\mathbb{R}^n)$ the set of all $\phi \in C^\infty(\mathbb{R}^n)$ such that for any $k,\ h > 0$

$$(1.1) \qquad |\phi|_{k,h} = \sup_{\substack{x\in\mathbb{R}^n \\ \alpha\in\mathbb{N}_0^n}} \frac{|\partial^\alpha \phi(x)| \exp k|x|}{h^{|\alpha|}\alpha!} < \infty$$

where $\mathbb{N}_0$ is the set of all nonnegative integers. The topology in $\mathcal{G}$ defined by the semi-norms in (1.1) makes $\mathcal{G}$ a Fréchet space. In fact, it is the projective limit topology over all $h > 0$ and $k > 0$.

REMARK. Replacing the quantifier *any* by *some* in the above definition we obtain the space $\mathcal{F}$ of test functions for the Fourier hyperfunctions. See [7] for more details.

In this paper we will construct the Gelfand triple based on the above space $\mathcal{G}'$ and investigate many properties of this space, which are natural extension of Paley–Wiener theorem and Bochner–Schwartz theorem.

2. Construction of Gelfand triples

The Hermite function of order $j,\ j \geq 0$, is defined by

$$e_j(x) = (\sqrt{\pi}2^n n!)^{-1/2} H_j(x) e^{-x^2/2},$$

where $H_j(x)$ is the Hermite polynomial of degree $j,\ j \geq 0$. It is easy to see that the set $\{e_j; j \geq 0\}$ is contained in $\mathcal{G}(\mathbb{R})$ and forms an orthonormal basis for $L^2(\mathbb{R})$. Let A be the Hermite operator and B be the following differential operators of infinite order, which are defined by

$$A = -\left(\frac{d}{dx}\right)^2 + x^2 + 1,$$

$$B = \sum_{k=0}^{\infty} \frac{1}{(2k)!} A^k.$$

Note that the operator B is formally equal to $\cosh\sqrt{A}$ and that e_j is an eigenfunction of A with eigenvalue $2j + 2$, i.e.,

$$Ae_j = (2j+2)e_j, \quad j \geq 0.$$

Therefore we obtain

$$Be_j = (\cosh\sqrt{2j+2})e_j, \quad j \geq 0.$$

For each $p \geq 0$ we define a norm on $\mathcal{G}(\mathbb{R})$ by

$$|f|_p = |B^p f|_0, \quad f \in \mathcal{G}(\mathbb{R})$$

where $|\cdot|_0$ is L^2-norm. Equivalently, this norm is given by

$$|f|_p = \left(\sum_{j=0}^{\infty}(\cosh\sqrt{2j+2})^{2p} < f, e_j >^2\right)^{1/2}$$

Let $\mathcal{G}_p(\mathbb{R})$ be the completion of $\mathcal{G}(\mathbb{R})$ with respect to the norm $|\cdot|_p$. Then $\mathcal{G}_p(\mathbb{R})$ becomes a Hilbert space and we have a natural inclusion relation;

$$\mathcal{G}_q \subset \mathcal{G}_p \subset L^2, \quad q \geq p \geq 0.$$

In fact, if $q \geq p \geq 0$ it follows

$$|f|_p \leq \delta^{q-p}|f|_q, \quad f \in \mathcal{G}_q$$

where $\delta = (\cosh\sqrt{2})^{-1}$.

Let $f_{p,q}; \mathcal{G}_q \to \mathcal{G}_p$ be the canonical injection. Then we have the following

LEMMA 2.1. *We retain the same notations and assumptions as above. The natural injection $f_{p,p+r}; \mathcal{G}_{p+r} \to \mathcal{G}_p$ is of Hilbert-Schmidt type for all $p \geq 0$.*

PROOF. Note that $\{(\cosh\sqrt{2j+2})^{-(p+r)}e_j\}_{j=0}^{\infty}$ is a complete orthonormal basis for $\mathcal{G}_{p+r}$. Since we can easily obtain that

$$\begin{aligned}\sum_{j=0}^{\infty}|(\cosh\sqrt{2j+2})^{-(p+r)}e_j|_p^2 &= \sum_{j=0}^{\infty}(\cosh\sqrt{2j+2})^{-2p} \\ &\leq \sum_{j=0}^{\infty} e^{-2(2j+2)p} < \infty,\end{aligned}$$

$f_{p,p+r}$ is of Hilbert-Schmidt type. $\square$

It follows from Lemma 2.1 that $\mathcal{G}(\mathbb{R})$ can be considered as the projective limit of the family $\{\mathcal{G}_p(\mathbb{R}) \,|\, p \geq 0\}$ and $\mathcal{G}(\mathbb{R})$ becomes a nuclear Fréchet space. Thus we have

$$\mathcal{G}'(\mathbb{R}) = \cup_{p\geq 0}\mathcal{G}'_p(\mathbb{R}) \cong \operatorname*{ind\,lim}_{p\to\infty} \mathcal{G}'_p(\mathbb{R}).$$

For $p \geq 0$, let $\mathcal{G}_{-p}$ be the completion of L^2 with respect to the norm

$$|f|_{-p} = |B^{-p}f|_0 = \left(\sum_{j=0}^{\infty}(\cosh\sqrt{2j+2})^{-2p}\langle f, e_j\rangle^2\right)^{1/2}, \quad f \in L^2.$$

Then the dual space $\mathcal{G}'_p(\mathbb{R})$ of $\mathcal{G}(\mathbb{R})$ is given by $\mathcal{G}_{-p}(\mathbb{R})$. Moreover, we have the following inclusion maps:

$$\mathcal{G}(\mathbb{R}) \subset \mathcal{G}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{G}'_p(\mathbb{R}) \subset \mathcal{G}'(\mathbb{R}), \quad p \geq 0.$$

Comparing this triple with the Gelfand triple of Hida based on the Schwartz space we can easily see that

$$\mathcal{G}_p(\mathbb{R}) \subset \mathcal{S}_p(\mathbb{R}) \subset L^2(\mathbb{R}) \subset \mathcal{S}'_p(\mathbb{R}) \subset \mathcal{G}'_p(\mathbb{R}).$$

3. Construction of generalized functionals

By the Bochner–Minors theorem, there exists a unique probability measure μ on $\mathcal{G}'(\mathbb{R})$ such that

$$\int_{\mathcal{G}'(\mathbb{R})} e^{i\langle x,f\rangle}\,d\mu(x) = \exp\left(-|f|_0^2/2\right), \quad f \in \mathcal{G},$$

where $|\cdot|_0$ is the L^2-norm.

By the Wiener–Itô theorem, $\phi \in L^2(\mathcal{G}'(\mathbb{R}), \mu) = (L^2)$ can be represented by

$$\phi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_{\mathbb{C}}^{2\widehat{\otimes} n} \tag{3.1}$$

where I_n is the multiple Wiener integral of order n and $L_{\mathbb{C}}^{2\widehat{\otimes}n}$ denotes the n-th symmetric tensor product of the complexification of L^2. Moreover, for $\phi \in (L^2)$ we have

$$||\phi||_0^2 = \sum_{n=0}^{\infty} n! |f_n|_0^2$$

where $||\cdot||_0$ stands for the norm of (L^2). We now define an operator $\Gamma(B)$ densely defined on (L^2) by

$$\Gamma(B)\phi = \sum_{n=0}^{\infty} I_n(B^{\otimes n} f_n).$$

The Sobolev spaces on $\mathcal{G}'(\mathbb{R})$ are constructed in a similar way as in §2. For $p \geq 0$, we define a norm $||\cdot||_p$ by

$$||\varphi||_p = ||\Gamma(B)^p \varphi||_0.$$

It is easy to see that $||\varphi||_0 \leq ||\varphi||_p$, $p \geq 0$ and so we define the Hilbert space $(\mathcal{G})_p$ for $p \geq 0$ by

$$(\mathcal{G})_p = \{\varphi \in (L^2); ||\varphi||_p < \infty\}.$$

We obviously have

$$(\mathcal{G})_q \subseteq (\mathcal{G})_p \subseteq (L^2), \quad q \geq p \geq 0.$$

We write $(\mathcal{G}) = \bigcap_{p \geq 0} (\mathcal{G})_p$ and introduce on $(\mathcal{G})$ the projective limit topology of the family of Hilbert spaces $\{(\mathcal{G})_p | p \geq 0\}$. On the other hand, we have $||\varphi||_{-p} \leq ||\varphi||_2$ and so we define the Hilbert space $(\mathcal{G})_{-p}$ for $p \geq 0$ by

$$(\mathcal{G})_{-p} = \text{completion of} \quad (L^2) \quad \text{with respect to} \quad ||\cdot||_{-p}.$$

It is easy to see that the dual space $(\mathcal{G})_p^*$ of $(\mathcal{G})_p$ is given by $(\mathcal{G})_{-p}$. Let $(\mathcal{G})^*$ denote the dual of $(\mathcal{G})$. Then we can easily see that

$$(\mathcal{G})^* = \bigcup_{p \geq 0} (\mathcal{G})_{-p},$$

and introduce on $(\mathcal{G})^*$ the inductive limit topology of the family of spaces $\{(\mathcal{G})_{-p} | p \geq 0\}$. Thus we construct the triple

$$(\mathcal{G}) \subset (L^2) \subset (\mathcal{G})^*.$$

Also, we can easily see that

$$(\mathcal{G}) \subset (\mathcal{S}) \subset (L^2) \subset (\mathcal{S})^* \subset (\mathcal{G})^*.$$

EXAMPLE. The Wick exponential

$$\Gamma_f(x) =: e^{\langle x,f\rangle} := \exp\left(\langle x,f\rangle - |f|_0^2/2\right), \quad f \in \mathcal{G},\ x \in \mathcal{G}'(\mathbb{R})$$

is contained in $(\mathcal{G})$. For, the functional Γ_f has the following decomposition

$$\Gamma_f = \sum_{n=0}^{\infty} I_n\left(\frac{1}{n!} f^{\otimes n}\right)$$

as in [6] and for any $p \geq 0$

$$||\Gamma_f||_p^2 = \sum_{n=0}^{\infty} \frac{1}{n!} |B^p f|_0^{2n} = \sum_{n=0}^{\infty} \frac{1}{n!} |f|_p^{2n} = e^{|f|_p^2} < \infty.$$

Since $(\mathcal{G}) \subset (L^2)$, each $\phi \in (\mathcal{G})$ also admits an expression as in (3.1). It is then important to know when $\phi \in (L^2)$ belongs to $(\mathcal{G})$. In this connection we have the following analogs of the representation theorems in [6].

THEOREM 3.1. *Let $\phi \in (L^2)$ be expressed as*

$$\phi = \sum_{n=0}^{\infty} I_n(f_n), \quad f_n \in L_{\mathbb{C}}^{2\widehat{\otimes} n}.$$

Then $\phi \in (\mathcal{G})$ if and only if $f_n \in \mathcal{G}_{\mathbb{C}}^{\widehat{\otimes} n}$ for all $n = 0, 1, \cdots$, and $\sum_{n=0}^{\infty} n!\, |f_n|_p^2 < \infty$ for all $p \geq 0$.

THEOREM 3.2. *For each $\Phi \in (\mathcal{G})^*$ there exists a unique sequence $\{F_n\}_{n=0}^{\infty}$, $F_n \in (\mathcal{G}_{\mathbb{C}}^{\otimes n})_{sym}^*$ such that*

$$\langle\langle \Phi, \phi \rangle\rangle = \sum_{n=0}^{\infty} n! \langle F_n, f_n \rangle, \tag{2.2}$$

for all $\phi \in (\mathcal{G})$ with $\phi = \sum_{n=0}^{\infty} I_n(f_n)$ where $f_n \in \mathcal{G}_{\mathbb{C}}^{\widehat{\otimes} n}$ with $\sum_{n=0}^{\infty} n! |f_n|_p^2 < \infty$ for all $p \geq 0$.

Conversely, for a sequence $\{F_n\}_{n=0}^{\infty}$, $F_n \in (\mathcal{G}_{\mathbb{C}}^{\otimes n})_{sym}^*$ with $\sum_{n=0}^{\infty} n! |F_n|_{-p}^2 < \infty$ for $p \geq 0$, a generalized functional $\Phi \in (\mathcal{G})^*$ is defined by (2.2).

In this case,

$$||\Phi||_{-p} = \sum_{n=0}^{\infty} n! |F_n|_{-p}^2.$$

By Example in §2 we can define S-transform on $(\mathcal{G})^*$.

DEFINITION 3.3. The S-transform of $\Phi \in (\mathcal{G})^*$ is a function on $\mathcal{G}$ defined by

$$(S\Phi)(f) = \langle \Phi, : \exp\langle \cdot, f \rangle : \rangle, \quad f \in \mathcal{G}$$

Finally, as a generalization of the results in Kuo [8] we can obtain the characterization theorems for white noise hyperfunctions in terms of their S-transform, which is an analog of Paley-Wiener theorem. Also, we can obtain the characterization theorems for positive white noise hyperfunctions, which is an analog of Bochner-Schwartz theorem in Fourier analysis.

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References

[1] S.-Y. Chung and D. Kim, *Distributions with exponential growth and Bochner-Schwartz theorem for Fourier hyperfunctions*, Publ. RIMS, Kyoto Univ. **31** (1995), 829–845.

[2] J. Chung, S.-Y. Chung and D. Kim, *A characterization for Fourier hyperfunctions*, Publ. RIMS, Kyoto Univ. **30** (1994), 203–208.

[3] ———, *Positive definite hyperfunctions*, Nagoya Math. J. **140** (1995), 139–149.

[4] S.-Y. Chung, D. Kim and S. K. Kim, Solvability of Mizohata and Lewy operators, J. Fac. Sci. Univ. Tokyo, Sect. IA **40** (1993), 63–71.

[5] ———, *Structure of the extended Fourier hyperfunctions*, Japan. J. Math. **19** (1993), 217–226.

[6] T. Hida, H.-H. Kuo, J. Potthoff and L. Streit, *White noise: An infinite dimensional calculus*, Kluwer, Dordrecht, 1993.

[7] K. H. Kim, S.-Y. Chung and D. Kim, *Fourier hyperfunctions as the boundary values of smooth solutions of heat equations*, Publ. RIMS, Kyoto Univ. **29** (1993), 287–300.

[8] H.-H. Kuo, *Lectures on white noise analysis*, Soochow J. Math. **18**, 229–300.

[9] N. Obata, *Elements of white noise calculus*, Lect. Notes in Math. Vol. 1577, Springer, Berlin-New York, 1995.

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