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ON THE EXISTENCE OF SOLUTIONS FOR SOME VARIATIONAL-LIKE INEQUALITIES

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ABSTRACT. In this paper we consider a kind of Minty's lemma for multifunctions in Banach spaces, and apply it to obtain existence theorems for two kinds of variational-like inequalities using the KKM-Fan theorem.

1. Introduction

In the last decade, there have been great developments in theory and application of (scalar) variational inequalities [9-12, 14-15] and vector variational inequalities [1-3, 7, 13]. Parida et al. [12], and Yang and Chen [15] studied the existence of solution of variational-like inequalities in $\mathbb{R}^n$ and showed a relationship between variational-like inequality problems and convex programming as well as with complementarity problems. Further, the existence of the solution of variational-like inequalities have been studied in reflexive Banach spaces and topological vector spaces with or without convexity assumptions by Siddiqi et al. [14].

On the other hand, the concept of Hausdorff metric have been useful and important in dealing with distance between sets, in particular, compact sets. Nadler [8] showed a remark for the Hausdorff distance between compact sets.

In this paper, first we obtain a kind of Minty's lemma in [4] for multifunctions in Banach spaces by using Nadler's results for compact sets. And then by using our result with KKM-Fan theorem we obtain existence theorems for two kinds of variational-like inequalities under the

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continuity of a nonempty compact-valued multifunctions. The continuity of a multifunction $T : X \to 2^{X^*}$ can be compared with the usual condition that a map from K to $\mathbb{R}$ defined by

$$x \longmapsto \langle T(x), \theta(y,x)\rangle + \eta(x,y)$$

is upper semicontinuous for each $y \in K$, where $T : K \to X^*$, $\theta : K \times K \to X$ and $\eta : K \times K \to \mathbb{R}$ are single-valued functions in real reflexive Banach spaces [5]. Of course, our method can be extended to the existing variational-like inequality problems.

2. A kind of Minty's lemma

First, let us give the following lemma in [8].

LEMMA 2.1. *Let $(X, ||\cdot||)$ be a normed vector space and let $H : 2^X \times 2^X \to [0,\infty)$ be the Hausdorff metric induced by the norm $||\cdot||$, which is defined by*

$$H(A,B) = \max\left(\sup_{x\in A}\inf_{y\in B}||x-y||, \sup_{y\in B}\inf_{x\in A}||x-y||\right)$$

for sets A and B in X.

Suppose that A and B are compact sets in X. Then for each $x \in A$, there exists $y \in B$ such that

$$||x-y|| \le H(A,B).$$

Now we obtain a kind of Minty's lemma for the case of multifunctions.

We denote by $\langle y, x\rangle$ the duality pairing between elements $y \in X^*$ and $x \in X$.

THEOREM 2.2. *Let X be a real Banach space, and K a nonempty convex subset of X. Let $T : K \to 2^{X^*}$ be a nonempty compact-valued multifunction such that for any x, $y \in K$,*

$$H(T(x+\lambda(y-x)), T(x)) \longrightarrow 0 \quad \text{as} \quad \lambda \to 0^+,$$

where H is the Hausdorff metric defined on the collection of all nonempty norm-closed subsets of X^*, and $\eta : K \times K \to X$ is an operator. Suppose that the following hold:

(i) $\langle t, \eta(y,y)\rangle \geq 0$ *for each* $x, y \in K$ *and* $t \in T(y)$,

(ii) *the operator*

$$x \longmapsto \eta(y,x)$$

of K into X is continuous for each $y \in K$,

(iii) *the map*

$$x \longmapsto \langle t, \eta(x,y)\rangle$$

of K into $\mathbb{R}$ is affine for each $y \in K$ and $t \in T(y)$,

(iv) *for each x, $y \in K$, the existence of $s \in T(x)$ such that*

$$\langle s, \eta(y,x)\rangle \geq 0$$

implies

$$\langle t, \eta(x,y)\rangle \leq 0$$

for any $t \in T(y)$.

Then the following are equivalent:

(a) *There exists an $x_0 \in K$ such that for each $y \in K$, there exists an $s_0 \in T(x_0)$ such that*

$$\langle s_0, \eta(y,x_0)\rangle \geq 0.$$

(b) *There exists an $x_0 \in K$ such that*

$$\langle t, \eta(x_0,y)\rangle \leq 0,$$

for all $y \in K$ and $t \in T(y)$.

PROOF. Suppose that there exists an $x_0 \in K$ such that for each $y \in K$, there exists $s_0 \in T(x_0)$ satisfying $\langle s_0, \eta(y,x_0)\rangle \geq 0$. Then it follows from (iv) that (b) holds.

Conversely, suppose that there exists an $x_0 \in K$ such that

$$\langle t, \eta(x_0,y)\rangle \leq 0$$

for all $y \in K$ and $t \in T(y)$. For any arbitrary $y \in K$, letting $y_\lambda = \lambda y + (1-\lambda)x_0$, $0 < \lambda < 1$, we have $y_\lambda \in K$ by the convexity of K. Hence for all $t_\lambda \in T(y_\lambda)$

$$\langle t_\lambda, \eta(x_0, y_\lambda)\rangle \leq 0. \tag{2.1}$$

By the affinity of the operator

$$x \longmapsto \langle t, \eta(x, y)\rangle ,$$

we have

$$\begin{aligned}\langle t_\lambda, \eta(y_\lambda, y_\lambda)\rangle &= \langle t_\lambda, \eta(\lambda y + (1-\lambda)x_0, y_\lambda)\rangle \\ &= \lambda \langle t_\lambda, \eta(y, y_\lambda)\rangle + (1-\lambda)\langle t_\lambda, \eta(x_0, y_\lambda)\rangle .\end{aligned}$$

Hence

$$\langle t_\lambda, \eta(y, y_\lambda)\rangle \geq 0. \tag{2.2}$$

In fact, suppose to the contrary that

$$\langle t_\lambda, \eta(y, y_\lambda)\rangle < 0.$$

Since

$$\langle t_\lambda, \eta(y_\lambda, y_\lambda)\rangle \geq 0$$

by (i), we have

$$(1-\lambda)\langle t_\lambda, \eta(x_0, y_\lambda)\rangle = \langle t_\lambda, \eta(y_\lambda, y_\lambda)\rangle - \lambda \langle t_\lambda, \eta(y, y_\lambda)\rangle > 0.$$

Thus

$$\langle t_\lambda, \eta(x_0, y_\lambda)\rangle > 0,$$

which contradicts (2.1). Hence

$$\langle t_\lambda, \eta(y, y_\lambda)\rangle \geq 0.$$

Since $T(y_\lambda)$ and $T(x_0)$ are compact, by Lemma 2.1, for each $t_\lambda \in T(y_\lambda)$ we can find an $s_\lambda \in T(x_0)$ such that

$$||t_\lambda - s_\lambda|| \leq H(T(y_\lambda), T(x_0)).$$

Since $T(x_0)$ is compact, without loss of generality, we can assume that for some $s_0 \in T(x_0)$, $s_\lambda \to s_0 \in T(x_0)$ as $\lambda \to 0^+$. Moreover we have

$$\begin{aligned}||t_\lambda - s_0|| &\le ||t_\lambda - s_\lambda|| + ||s_\lambda - s_0|| \\ &\le H(T(y_\lambda), T(x_0)) + ||s_\lambda - s_0||.\end{aligned}$$

Since $H(T(y_\lambda), T(x_0)) \to 0$ as $\lambda \to 0^+$, $t_\lambda \to s_0$. By assumption (**i**)

$$\eta(y, y_\lambda) \longrightarrow \eta(y, x_0) \quad \text{as} \quad \lambda \to 0^+.$$

Moreover we have

$$\begin{aligned}&|| \langle t_\lambda, \eta(y, y_\lambda)\rangle - \langle s_0, \eta(y, x_0)\rangle || \\ =&|| \langle t_\lambda, \eta(y, y_\lambda)\rangle - \langle s_0, \eta(y, y_\lambda)\rangle + \langle s_0, \eta(y, y_\lambda)\rangle - \langle s_0, \eta(y, x_0)\rangle || \\ \le&|| \langle t_\lambda - s_0, \eta(y, y_\lambda)\rangle || + || \langle s_0, \eta(y, y_\lambda) - \eta(y, x_0)\rangle || \\ \le&||t_\lambda - s_0|| ||\eta(y, y_\lambda)|| + ||s_0|| ||\eta(y, y_\lambda) - \eta(y, x_0)||.\end{aligned}$$

Hence

$$\langle t_\lambda, \eta(y, y_\lambda)\rangle \longrightarrow \langle s_0, \eta(y, x_0)\rangle \quad \text{as} \quad \lambda \to 0^+.$$

So, it follows from (2.2) that

$$\langle s_0, \eta(y, x_0)\rangle \ge 0 \quad \text{for all} \quad y \in K.$$

This completes the proof. □

REMARK. Theorem 2.2 can be said to be a partial generalization of the following classical Minty's lemma by weakening the monotonicity for single-valued mapping defined in a reflexive Banach space.

COROLLARY 2.3. *Let X be a real reflexive Banach space, K a nonempty closed convex subset of X and X^* the dual of X. Let $T : K \to X^*$ be a monotone and hemicontinuous operator. Then the following are equivalent:*

(a) *There exists an $x_0 \in K$ such that*

$$\langle T(x_0), y - x_0\rangle \ge 0 \quad \textit{for all} \quad y \in K.$$

(b) *There exists an $x_0 \in K$ such that*

$$\langle T(y), y - x_0\rangle \ge 0 \quad \textit{for all} \quad y \in K.$$

3. Existence theorem

Now we will apply Theorem 2.2 to show the existence of solutions for two kinds of variational-like inequalities. The following useful and important KKM-Fan theorem will be used in the proof.

DEFINITION 3.1. Let K be a subset of a topological vector space X. Then a multifunction $F : K \to 2^X$ is called Knaster-Kuratowski-Mazurkiewicz (in short, KKM)-multifunction if for each nonempty finite subset N of K, $co\ N \subset F(N)$, where co denotes the convex hull and $F(N) = \bigcup\{F(x) : x \in N\}$.

THEOREM 3.1 ([6]). *Let K be an arbitrary nonempty subset of a Hausdorff topological vector space. Let $F : K \to 2^X$ be a KKM-multifunction such that $F(x)$ is closed for all $x \in K$ and is compact for at least one $x \in K$.*

Then

$$\bigcap_{x \in K} F(x) \neq \phi.$$

THEOREM 3.2. *Let X be a real Banach space and K a nonempty compact convex subset of X.*

Assume that $T : K \to 2^{X^*}$ be a multifunction and $\eta : K \times K \to X$ an operator satisfying the following conditions:

(i) $\langle t, \eta(y,y)\rangle \geq 0$ *for each x, $y \in K$ and $t \in T(y)$,*

(ii) *the operator*

$$x \longmapsto \eta(x,y)$$

of K into X is continuous for each $y \in K$,

(iii) *the map*

$$x \longmapsto \langle t, \eta(x,y)\rangle$$

of K into $\mathbb{R}$ is affine for each $y \in K$ and $t \in T(y)$,

(iv) *for each x, $y \in K$, the existence of $s \in T(x)$ such that*

$$\langle s, \eta(y,x)\rangle \geq 0$$

implies

$$\langle t, \eta(x,y)\rangle \leq 0$$

for any $t \in T(y)$.

Then there exists a solution $x_0 \in K$ *of the following variational-like inequality:*
find $x_0 \in K$ *such that*

$$\langle t, \eta(x_0, y)\rangle \leq 0$$

for all $y \in K$ *and* $t \in T(x_0)$.

Moreover, if T *is a nonempty compact-valued multifunction such that for any* $x, y \in K$, $H(T(x+\lambda(y-x)), T(x)) \to 0$ *as* $\lambda \to 0^+$, *where* H *is the Hausdorff metric defined on the collection of all nonempty norm-closed subsets of* X^* *and the operator* $x \mapsto \eta(y, x)$ *of* K *into* X *is continuous for each* $y \in K$, *then there exists a solution* $x_0 \in K$ *of the following variational-like inequality:*
find $x_0 \in K$ *such that for each* $y \in K$, *there exists an* $s_0 \in T(x_0)$ *such that*

$$\langle s_0, \eta(y, x_0)\rangle \geq 0.$$

PROOF. Define a multifunction $F_1 : K \to 2^K$ by

$$F_1(y) = \{x \in K : \text{there exists } s \in T(x) \text{ such that } \langle s, \eta(y, x)\rangle \geq 0\}$$

for each $y \in K$.

Note that $F_1(y) \neq \phi$ for each $y \in K$, since $y \in F_1(y)$.

Then F_1 is a KKM multifunction on K.

In fact, suppose that $N = \{x_1, x_2, \cdots, x_n\} \subset K$, $\sum_{i=1}^{n} \alpha_i = 1$, $\alpha_i \geq 0$, $i = 1, 2, \cdots, n$ and $x = \sum_{i=1}^{n} \alpha_i x_i \notin F_1(N)$.

Then for any $s \in T(x)$,

$$\langle s, \eta(x_i, x)\rangle < 0,$$

$i = 1, 2, \cdots, n$. Thus we have

$$\left\langle s, \eta\big(x_j, \sum_{i=1}^{n} \alpha_i x_i\big)\right\rangle < 0,$$

for each $j = 1, 2, \cdots, n$. By the affinity of the operator

$$x \longmapsto \langle s, \eta(x, y) \rangle ,$$

it follows that

$$\begin{aligned} &\left\langle s, \eta\Big(\sum_{j=1}^{n} \alpha_j x_j, \sum_{i=1}^{n} \alpha_i x_i\Big) \right\rangle \\ &= \sum_{j=1}^{n} \alpha_j \left\langle s, \eta\Big(x_j, \sum_{i=1}^{n} \alpha_i x_i\Big) \right\rangle \\ &< 0, \end{aligned}$$

which contradicts (i). Hence F_1 is a KKM multifunction on K. Define a multifunction $F_2 : K \to 2^K$ by

$$F_2(y) = \{x \in K : \langle t, \eta(x, y) \rangle \le 0 \quad \text{for all} \quad t \in T(y)\},$$

then by (iv) $F_1(y) \subset F_2(y)$ for each $y \in K$. Therefore F_2 is also a KKM multifunction on K.

Now we show that $F_2(y)$ is closed. Let $\{x_n\}$ be a sequence in $F_2(y)$ converging to $x_0 \in K$. Then we have

$$\langle t, \eta(x_n, y) \rangle \le 0 \quad \text{for all} \quad t \in T(y). \tag{2.4}$$

By (i)

$$\eta(x_n, y) \longrightarrow \eta(x_0, y)$$

and hence for all $t \in T(y)$,

$$\langle t, \eta(x_n, y) \rangle \longrightarrow \langle t, \eta(x_0, y) \rangle \quad \text{as} \quad n \to \infty.$$

Letting $n \to \infty$, so

$$\langle t, \eta(x_0, y) \rangle \le 0$$

and hence $F_2(y)$ is closed. Since K is compact, so is $F_2(y)$ for all $y \in K$. Hence by the KKM-Fan theorem

$$\bigcap_{y \in K} F_2(y) \ne \phi,$$

i.e., there exists an $x_0 \in K$ such that

$$\langle t, \eta(x_0, y)\rangle \le 0$$

for any $y \in K$ and any $t \in T(y)$.

Moreover, if T is a nonempty compact-valued multifunction such that for any x, $y \in K$,

$$H(T(x + \lambda(y - x)), T(x)) \longrightarrow 0 \quad \text{as} \quad \lambda \to 0^+$$

and the operator $x \mapsto \eta(y, x)$ of K into X is continuous for each $y \in K$, then it follows from Theorem 2.2 that there exists an $x_0 \in K$ such that for each $y \in K$, there exists an $s_0 \in T(x_0)$ such that

$$\langle s_0, \eta(y, x_0)\rangle \ge 0. \qquad \square$$

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