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A NOTE ON CONVERTIBLE (0,1) MATRICES II

SI-JU KIM AND TAEG-YOUNG CHOI

ABSTRACT. Let A be an $n \times n$ (0,1) matrix. Let $f(A)$ denote the smallest nonnegative integer k such that $perA[\alpha|\beta] > 0$ and $A(\alpha|\beta)$ is permutation equivalent to a lower triangular matrix for some α, $\beta \in Q_{k,n}$. In this case $f(A)$ is called the feedback number of A. In this paper, feedback numbers of some maximal convertible (0,1) matrices are studied.

1. Introduction

Let $A = [a_{ij}]$ be any real matrix of order n. The permanent of A is defined by

$$perA = \sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)},$$

where S_n denotes the set of permutations of $1, 2, 3, \cdots, n$. A nonnegative $n \times n$ matrix A is called *convertible* if there exists a $(1, -1)$ matrix H such that $perA = \det(H \circ A)$ where $H \circ A$ denotes the Hadamard product of H and A. A square convertible $(0, 1)$ matrix is called *maximal* if replacing any zero entry with a 1 results in a non-convertible matrix.

For matrices A, B of the same size, A is said to be *permutation equivalent* to B, denoted by $A \sim B$, if there are permutation matrices P, Q such that $PAQ = B$. An $n \times n$ matrix is called *partly decomposable* if it contains a $t \times (n-t)$ zero submatrix for some positive integer t. Square matrices which are not partly decomposable are called *fully indecomposable.*

For positive integers k and n with $k \leq n$, let $Q_{k,n}$ denote the set of all strictly increasing k-sequences from $\{1, 2, \cdots, n\}$. For $\alpha \in Q_{k,n}$ and

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$\beta \in Q_{l,n}$, let $A(\alpha|\beta)$ denote the $(n-k) \times (n-l)$ submatrix obtained from an $n \times n$ matrix A by deleting rows α and columns β and let $A[\alpha|\beta]$ denote the matrix complementary to $A(\alpha|\beta)$ in A. Let A be an $n \times n$ (0,1) matrix. Let $f(A)$ denote the smallest nonnegative integer k such that $perA[\alpha|\beta] > 0$ and $A(\alpha|\beta)$ is permutation equivalent to a lower triangular matrix for some $\alpha,\ \beta \in Q_{k,n}$. In this case $f(A)$ is called the *feedback number* of A. In [3, 4, 5, 6, 7 and 8], the authors investigated some properties of maximal convertible matrices. In this paper, feedback numbers of some maximal convertible (0,1) matrices are investigated. Let $J_{n\times m}$ denote the $n \times m$ matrix all of whose entries are 1 and let I_n be the identity matrix of order n.

2. Main results

Before we present our results we restate the following well-known lemmas. All maximal convertible matrices to consider through this paper are fully indecomposable.

LEMMA A [1, PROPOSITION 3.2]. *Let A be a maximal convertible matrix of order n whose first column is equal to $[1,1,0,\cdots,0]^T$. Then the first two rows of A are identical and the matrix obtained from A by deleting row 1 and column 1 is a maximal convertible matrix.*

LEMMA B [3, THEOREM 5]. *A square submatrix of a convertible $(0,1)$ matrix is convertible if its complementary submatrix has positive permanent.*

Let A be a (0,1) matrix of size n. Then there always exist $\alpha, \beta \in Q_{n-k,n}$ such that $perA[\alpha|\beta] \leq 1$ and $perA(\alpha|\beta) \geq 1$ for some nonnegative integer k. Here we can take $A[\alpha|\beta]$ as a triangular matrix. It is easy to show that $f(A)$ runs over all integers in $\{0, 1, \cdots, n-1\}$ for an $n \times n$ (0,1) matrix A. Moreover for an $n \times n$ (0,1) matrix A, $f(A) = n-1$ if and only if $A = J_{n\times n}$. Let $\nu(A)$ denote the number of zero entries of a matrix A. Gibson [2] proved that for any $n \times n$ convertible (0,1) matrix A with $perA > 0$, $\nu(A) \geq (n^2 - 3n + 2)/2$. Therefore $f(A) < n-1$ for any $n \times n$ convertible (0,1) matrix A. It is hard to classify convertible matrix A with $f(A) = k$ for any possible integer k. However we can study maximal convertible matrix A with $f(A) = 1, 2$ and we propose a problem of feedback numbers of convertible (0,1) matrices.

Let A be a (0,1) matrix of size n. If $per A > 0$, $f(A) = 0$ if and only if $per A = 1$, i.e., A is permutation equivalent to a triangular matrix with 1's in the n main diagonal positions and with 0's above the main diagonal.

Let $T_n = [t_{ij}]$ denote the $n \times n$ $(0,1)$ matrix with $t_{ij} = 0$ if and only if $j > i+1$. Then we have the following condition for $f(A) = 1$.

THEOREM 2.1. *Let A be a maximal convertible matrix. Then $A \sim T_n$ if and only if $f(A) = 1$.*

PROOF. Let $A \sim T_n$. Clearly $f(T_n) = 1$ and hence $f(A) = 1$.

Conversely let $f(A) = 1$. Without loss of generality, we may assume that $A = [a_{ij}]$ is of the form

$$A = \begin{pmatrix} 1 & a_{12} & \cdots & \cdots & a_{1n} \\ a_{21} & a_{22} & & & \\ \vdots & \vdots & \ddots & O & \\ \vdots & \vdots & * & \ddots & \\ a_{n1} & a_{n2} & \cdots & \cdots & a_{nn} \end{pmatrix}.$$

Since A is fully indecomposable, $a_{1n} = a_{nn} = 1$. By Lemma A, $A(n|n)$ is fully indecomposable and hence $a_{1,n-1} = a_{n-1,n-1} = 1$. Applying Lemma A continuously, we have $a_{11} = a_{12} = \cdots = a_{1n} = a_{22} = a_{33} = \cdots = a_{nn} = 1$. Since A is maximal convertible, $a_{n1} = \cdots = a_{nn} = 1$ by Lemma A. Since $A(n|n)$ is also maximal convertible, $a_{n-1,1} = \cdots = a_{n-1,n-1} = 1$. Continuing this process, $a_{i1} = \cdots = a_{ii} = 1$ for all $i = 2, 3, \cdots, n$. Hence $A \sim T_n$. □

Let $T_{n-1} = [\mathbf{t}_1, \cdots, \mathbf{t}_{n-1}]$. For $k = 1, 2, \cdots, n-1$, let

$$V_{n,k} = \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \mathbf{t}_k & \mathbf{t}_1 & \cdots & \mathbf{t}_{k-1} & \mathbf{t}_k & \mathbf{t}_{k+1} & \cdots & \mathbf{t}_{n-1} \end{pmatrix}.$$

$V_{n,k}$ is called the k-th column expansion of T_{n-1}. Let the $n \times n$ matrix $W_{n,k}^T$ be the k-th column expansion of T_{n-1}^T. $W_{n,k}$ is called the k-th row expansion of T_{n-1}. A matrix is called an *expansion* of a (0,1) matrix A

if it is a row expansion or a column expansion of A. Then $V_{n,k}$ and $W_{n,k}$ are maximal convertible for $n \geq 3$ and for $k = 1, 2, \cdots, n-1$ ([3]). Note that

$$f(V_{n,k}) = \begin{cases} 1, & \text{for } n = 3 \text{ or } k = 1, 2 \\ 2, & \text{for } n \geq 4 \text{ and } k = 3, \cdots, n-1 \end{cases}$$

and

$$f(W_{n,k}) = \begin{cases} 1, & \text{for } n = 3 \text{ or } k = n-2, n-1 \\ 2, & \text{for } n \geq 4 \text{ and } k = 1, 2, \cdots, n-3. \end{cases}$$

Since an $n \times n$ maximal convertible matrix containing T_{n-1} as a submatrix is an expansion of T_{n-1} [7], we have following Corollary.

COROLLARY 2.2. *Let A be an $n \times n (n \geq 4)$ maximal convertible matrix containing a maximal convertible matrix B of size $n-1$ with $f(B) = 1$. Then $f(A) = 2$ if and only if $A \sim V_{n,k}$ for some $k \in \{3, \cdots, n-1\}$ or $A \sim W_{n,k}$ for some $k \in \{1, 2, \cdots, n-3\}$.*

An $n \times n$ nonnegative matrix A is *doubly indecomposable* if $per A(\alpha|\beta) > 0$ for all $\alpha,\ \beta \in Q_{2,n}$. Notice that doubly indecomposability implies fully indecomposability. From the definition of the doubly indecomposability and Lemma B we have

LEMMA 2.3. *Let A be a doubly indecomposable convertible $(0,1)$ matrix. Then A has no $J_{2\times3}$ or $J_{3\times2}$ as submatrices.*

LEMMA 2.4. *Let A be a doubly indecomposable maximal convertible matrix. Then every row (column) of A has at least three 1's.*

PROOF. Let A be a doubly indecomposable maximal convertible matrix. If A has a row with exactly two 1's, then A has two identical columns (cf. [1]). Since A is fully indecomposable, A contains a $J_{2\times3}$ as a submatrix. This is contradiction to Lemma 2.2. Similarly the result holds for columns. □

Let $P_n = [p_{ij}]$ be the permutation matrix of order n such that $p_{ij} = 1$ if and only if $(i,j) \in \{(1,2), (2,3), \cdots, (n-1,n), (n,1)\}$. Then

$$D_n = \begin{pmatrix} 1 & & & \\ \vdots & & I_{n-1} + P_{n-1} & \\ 1 & & & \\ 0 & 1 & \cdots & 1 \end{pmatrix}$$

is a doubly indecomposable maximal convertible matrix ([8]).

THEOREM 2.5. *Let A be a doubly indecomposable maximal convertible matrix. If $f(A) = 2$ and A has a row with $n-1$ nonzero entries, then $A \sim D_n$.*

PROOF. Without loss of generality, we may assume that $A = [a_{ij}]$ is of the form

$$A = \begin{pmatrix} & & a_{13} & & & \\ & & & \ddots & O & \\ & * & & * & \ddots & \\ & & & & & a_{n-2,n} \\ 1 & p & & * & & \\ q & 1 & & * & & \end{pmatrix}.$$

By Lemma 2.3,

$$A = \begin{pmatrix} 1 & 1 & 1 & & & \\ & & & \ddots & O & \\ & * & & * & \ddots & \\ & & & & & 1 \\ 1 & p & & * & & 1 \\ q & 1 & & * & & 1 \end{pmatrix}.$$

By Lemma 2.2, we have $p = q = 0$, $p = 1$ and $q = 0$, or $p = 0$ and $q = 1$.

Case 1. $p = 1$ and $q = 0$.

Since one of the last three rows of A has $n-1$ nonzero entries by hypothesis, permuting first two columns and last three columns properly, we may assume that A is one of the forms

$$\begin{pmatrix} 1 & 1 & 1 & & & & \\ & & & \ddots & O & & \\ & * & & * & \ddots & & \\ & & & & & & 1 \\ 1 & 1 & & * & & & 1 \\ 0 & 1 & 1 & \cdots & & \cdots & 1 \end{pmatrix} \text{ or } \begin{pmatrix} 1 & 1 & 1 & & & & & & \\ & & & \ddots & & & & & \\ & & & & \ddots & & O & & \\ & & & & & \ddots & & & \\ & * & & & * & & \ddots & & \\ & & & & & & & \ddots & \\ & & & & & & & & 1 \\ 1 & 1 & 1 & \cdots & 1 & 0 & 1 & \cdots & 1 \\ 0 & 1 & & * & & * & & & 1 \end{pmatrix}.$$

In the first case, $a_{n-1,3} = a_{n-1,4} = \cdots = a_{n-1,n-1} = 0$ by Lemma 2.2. Also $a_{n-2,n-1} = 1$ by Lemma 2.3. Then $a_{n-2,2} = \cdots = a_{n-2,n-2} = 0$ by Lemma 2.2. Also $a_{n-2,1} = a_{n-3,n-2} = 1$ by Lemma 2.3. Continuing this process, we have $a_{i2} = \cdots = a_{ii} = 0$ and $a_{i1} = a_{i,n-1} = 1$ for all $i = 2, 3, \cdots, n-2$. Hence $A \sim D_n$. Consider the second case. If $a_{n-1,3} \neq 0$, then $A[1, n-1|1,2,3] = J_{2\times 3}$. Hence $a_{n-1,3} = 0$ and $a_{n-1,1} = a_{n-1,2} = a_{n-1,4} = \cdots = a_{n-1,n} = 1$. By the similar method above, we have

$$A = \begin{pmatrix} 1 & 1 & 1 & & & & & \\ 1 & 0 & 1 & 1 & & O & & \\ 0 & 0 & 1 & 1 & \ddots & & & \\ 0 & 0 & 1 & 0 & \ddots & \ddots & & \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & \ddots & \\ 0 & 0 & 1 & 0 & \cdots & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 & \cdots & \cdots & 1 & 1 \\ 0 & 1 & 1 & 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \sim D_n.$$

Similarly we have the same result for $p = 0$ and $q = 1$.

Case 2. $p = q = 0$.

Since one of $(n-2)$th, $(n-1)$th or nth rows of A should have $n-1$ nonzero entries, permuting the first three columns and last three rows, we can return to the case 1. □

Notice that for A to have a row with $n-1$ nonzero entries is necessary in the Theorem 2.4. For example, let

$$A = \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

Then A is a doubly indecomposable maximal convertible matrix ([1]) and $f(A) = 2$, but

$$A \not\sim D_6.$$

Let $U_2 = T_2$ and let

$$U_n = \begin{pmatrix} 1 & \mathbf{a} \\ \mathbf{b} & U_{n-1} \end{pmatrix}$$

for $n \geq 3$, where

$$\mathbf{a} = \left(1, \frac{1+(-1)^n}{2}, 0, \cdots, 0\right), \ \mathbf{b} = \left(1, \frac{1-(-1)^n}{2}, 0, \cdots, 0\right)^T.$$

Then U_n is a maximal convertible matrix ([3]). It would be of interest to find the largest feedback number of $n \times n$ convertible (0,1) matrices. We propose a problem.

PROBLEM 2.6. *Is $f(A) \leq 6$ for any convertible $(0,1)$ matrix A? Does the equality hold if and only if $A \sim U_n$ for some n?*

At present, we cannot solve the problem. But we can show that $f(U_n) \leq 6$ for all $n \geq 2$.

LEMMA 2.7. *An $n \times n$ matrix A is permutation equivalent to a triangular matrix if and only if there exist positive integers i, j, k, l such that $A(i, k|j, l)$ is permutation equivalent to a triangular matrix and all entries except a_{il} and a_{kj} in the i-th row and j-th column of A are zero.*

PROOF. Necessity part is trivial. Conversely let $A = [a_{ij}]$. Without loss of generality, we may assume that $A(1,2|1,2)$ is an lower triangular matrix and $a_{12} = \cdots = a_{1n} = a_{21} = a_{23} = a_{24} = \cdots = a_{2n} = 0$. Permuting lines of A, we know that A is permutation equivalent to a lower triangular matrix. □

THEOREM 2.8. *$f(U_n) \leq 6$ for all $n \geq 2$.*

PROOF. Let $\alpha = (n-11, n-9, n-8, n-5, n-3, n-2)$ and $\beta = (n-11, n-8, n-7, n-5, n-2, n-1)$. Then $perU_n[\alpha|\beta] > 0$. It is easy to show that $U_n(n-11, n-10, n-9, n-8, n-6, n-5, n-4, n-3, n-2, n|n-11, n-10, n-8, n-7, n-6, n-5, n-4, n-2, n-1, n)$ is similar to a lower triangular matrix. Using Lemma 2.5, we can show that $U_n(\alpha|\beta)$ is permutation equivalent to a lower triangular matrix. Hence $f(U_n) \leq 6$. □

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Department of Mathematics Education
Andong University
Andong 760-749, Korea
E-mail: sjkim@anu.andong.ac.kr