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MODULES OF QUOTIENTS OVER COMMUTATIVE RINGS

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ABSTRACT. In this paper, we give an affirmative answer to the question raised in [5]; whether $\mathcal{L}((\mathcal{P}))$ is principal or not. Using this fact, we try to give concrete form of module of quotient with respect to a torsion theory determined by $\mathcal{L}((\mathcal{P}))$.

In [3] Goldman introduced the notion of modules of quotients of a ring with respect to an idempotent kernel functor, which is a generalization of the localization of a module with respect to a multiplicative subset of a commutative ring.

A functor σ on R-mod, the category of R-modules, is called an *idempotent kernel functor* if the following properties hold:

(1) For every R-module M, $\sigma(M)$ is a submodule of M.
(2) If $f : M' \longrightarrow M$ is a homomorphism, then $f(\sigma(M')) \subset \sigma(M)$ and $\sigma(f)$ is a restriction of f to $\sigma(M')$.
(3) If M' is a submodule of M, then $\sigma(M') = \sigma(M) \cap M'$.
(4) $\sigma(M/\sigma(M)) = 0$.

We say M is a *σ-torsion* (resp. *σ-torsion free*) R-module if $\sigma(M) = M$ (resp. $\sigma(M) = 0$).

And we remark that σ is an idempotent kernel functor if and only if the following holds:

If $0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$ is an exact sequence such that M' and M'' are σ-torsion R-modules, then M is also a σ-torsion R-module.

A nonempty set $\mathcal{L}$ of ideals of a ring R is called a *Gabriel filter* if and only if the following two conditions are satisfied:

(1) If $I \in \mathcal{L}$ and $a \in R$, then $(I : a) \in \mathcal{L}$.

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(2) If I is a left ideal of R for which there exists an element H of $\mathcal{L}$ satisfying $(I : a) \in \mathcal{L}$ for all $a \in H$, then $I \in \mathcal{L}$.

If $\mathcal{L}$ is a Gabriel filter of ideals of R, then the following conditions are satisfied:

(1) If $I \in \mathcal{L}$ and if H is an ideal of R containing I, then $H \in \mathcal{L}$.
(2) If I, $H \in \mathcal{L}$, then $I \cap H \in \mathcal{L}$.
(3) If I, $H \in \mathcal{L}$, then $IH \in \mathcal{L}$.

All rings in this paper are commutative with identity, and all modules are unital. Specially for a given set of prime ideals $\mathcal{P}$ of a ring R, we consider the corresponding Gabriel filter $\mathcal{L}((\mathcal{P}))$, concise definition shall come latter. And we will give an affirmative answer to the question, Whether $\mathcal{L}((\mathcal{P}))$ is principal or not." which was raised in the remark of [5].

Let R be a ring and P be a prime ideal of R. Then $S = R - P$ is a multiplicatively closed subset of R. The ring of quotients $S^{-1}R$ is called the *localization of R at P* and is denoted by R_P. If I is an ideal in R, then the ideal $S^{-1}I$ in R_P is denoted by I_P. First we consider a filter determined by a set of prime ideals of R.

LEMMA 1 [5]. *For each set $\mathcal{P}$ of prime ideals, the set*

$$\mathcal{L}(\mathcal{P}) = \{I \lhd R \,|\, I_P = R_P \quad \text{for all } P \in \mathcal{P}\}$$

is a Gabriel filter.

Following [1] or [2], we say that a Gabriel filter $\mathcal{L}$ is *principal filter* if each ideal I of $\mathcal{L}$ contains a principal ideal that is in $\mathcal{L}$, and the intersection of principal filter is again a Gabriel filter and called a *primal filter.*

PROPOSITION 2 [7, Proposition 15.1]. *There exists a ono-to-one correspondence between principal filters on R and subsets S on R satisfying the followings:*

(1) $1 \in S$.
(2) s_1, $s_2 \in S$ *implies* $s_1 s_2 \in S$.
(3) *If* $r \in R$ *and* $s \in S$, *then there exist* $r' \in R$ *and* $s' \in S$ *such that* $r's = rs'$.
(4) *If* a, $b \in R$ *and* $ab \in S$, *then* $a \in S$.

Usually S is called a saturated set if S satisfies the condition (4).

For any R-module M and a prime ideal P of R, let

$$M_{(P)} = \{m/s \mid s \text{ is a regular element not in } P \text{ and } m \in M\}.$$

Then $R_{(P)}$ is a localization of R and $M_{(P)}$ is an $R_{(P)}$-module. Here a regular element means a non-zero divisor in R.

As a special case of Lemma 1, we can get the following result which appear in [5].

LEMMA 3 [5, Lemma 2]. *Let $\mathcal{P}$ be a set of prime ideals of R and let*

$$\mathcal{L}((\mathcal{P})) = \{I \lhd R \mid I_{(P)} = R_{(P)} \quad \text{for all } P \in \mathcal{P}\}.$$

Then $\mathcal{L}((\mathcal{P}))$ is a Gabriel filter.

For convenience, we adopt the following notation in this section.
$C(P) = \{r \in R \mid r + P \text{ is a regular element in } R/P\}$.
If $\mathcal{P}$ is a set of prime ideals P in R, then we denote
$S(\mathcal{P}) = R - \bigcap_{P\in\mathcal{P}} P$ and
$C(\mathcal{P}) = \bigcap_{P\in\mathcal{P}} C(P)$.

REMARK 4. We list some properties on $S(\mathcal{P})$ and $C(\mathcal{P})$:

(1) If R is an integral domain, then $S(P) = C(P)$ for one prime ideal P in R.
(2) $C(\mathcal{P}) \subset S(\mathcal{P})$ where $\mathcal{P}$ is a set of prime ideals in R.
(3) $C(\mathcal{P})$ is a saturated set.
(4) In general, $S(\mathcal{P})$ is not equal $C(\mathcal{P})$.

PROOF. The statements (1), (2) and (3) are clear. To show (4), we give counter example. Let $R = \mathbb{Z}$ and $\mathcal{P}_n = \{2\mathbb{Z}, 3\mathbb{Z}, 5\mathbb{Z}, \cdots, k_n\mathbb{Z}\}$ where k_n is n-th prime in $\mathbb{Z}$. Then $S(\mathcal{P}_n) = \mathbb{Z} - \bigcap_{i=1}^{n} k_i\mathbb{Z}$ where k_i is i-th prime in $\mathbb{Z}$. Since $30 \notin 7\mathbb{Z}$, $30 = 2 \cdot 3 \cdot 5 \in S(\mathcal{P}_4)$. But $30 \notin \bigcap_{i=1}^{4} C(k_i\mathbb{Z})$. So, (4) holds. □

Now we consider following two sets of ideals in R, which appear in [5].

$$\begin{aligned}\mathcal{L}(C(\mathcal{P})) &= \{I \lhd R \mid C(P)^{-1}R = C(P)^{-1}I \quad \text{for each } P \in \mathcal{P}\} \\ &= \{I \lhd R \mid I_{(P)} = R_{(P)} \quad \text{for each } P \in \mathcal{P}\}\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}(S(\mathcal{P})) &= \{I \lhd R \mid S(P)^{-1}R = S(P)^{-1}I \quad \text{for each } P \in \mathcal{P}\}.\\ &= \{I \lhd R \mid I_P = R_P \quad \text{for each } P \in \mathcal{P}\}.\end{aligned}$$

Now we prove the main theorem:

THEOREM 5. *Let $\mathcal{P}$ be a set of prime ideals of R, then $\mathcal{L}(C(\mathcal{P}))$ is a principal filter. In particular, $\mathcal{L}(C(P))$ is a principal filter for a prime ideal P of R.*

PROOF. We can see that $C(\mathcal{P})$ is a multiplicatively closed set. Since R is a commutative ring, $C(\mathcal{P})$ is an Ore set. By Remark 4, $C(\mathcal{P})$ is a saturated set. Thus by Proposition 2, $\mathcal{L}(C(\mathcal{P}))$ is a principal filter. □

Moreover, we can see that this Gabriel filter is perfect. Let's denote $C(\mathcal{P}) = C$. Then by [7], $C^{-1}M \cong C^{-1}R \bigotimes_R M$ for each left R-module M. So, $\mathcal{L}(C(\mathcal{P}))$ is a perfect.
The following question was raised in [5] ; whether $\mathcal{L}(S(\mathcal{P}))$ is principal or not. Since R is a commutative ring, $S(\mathcal{P})$ is an Ore set. But $S(\mathcal{P})$ is not a saturated set in general (cf. [10]), thus we can say that $\mathcal{L}(S(\mathcal{P}))$ need not be a principal filter.

Let M be an arbitrary R-module. Let $\mathcal{L}$ be a Gabriel filter and σ be the corresponding idempotent kernel functor. By [7], we can define $M_{\mathcal{L}} = \varinjlim Hom_R(I, M/\sigma(M))$ where $I \in \mathcal{L}$.

One verifies that the ring structure of $R_{\mathcal{L}}$ and the module structure of $M_{\mathcal{L}}$ are given by the following pairing $M_{\mathcal{L}} \times R_{\mathcal{L}} \longrightarrow M_{\mathcal{L}}$:

$$\text{Let } x \in M_{\mathcal{L}} \text{ be represented by } \xi : J \longrightarrow M/\sigma(M) \text{ where } J \in \mathcal{L},$$
$$a \in R_{\mathcal{L}} \text{ be represented by } \alpha : I \longrightarrow R/\sigma(R) \text{ where } I \in \mathcal{L}.$$

ξ induces $J/\sigma(J) \to M/\sigma(M)$ and we have $J/\sigma(J) \hookrightarrow R/\sigma(R)$ by left exactness of σ; $xa \in M_{\mathcal{L}}$ is represented by

$$\alpha^{-1}(J/\sigma(J)) \longrightarrow J/\sigma(J) \longrightarrow M/\sigma(M).$$

PROPOSITION 6 [7, Proposition 15.2]. *If $\mathcal{L}$ is a principal filter on R, S is the subset of R corresponding to $\mathcal{L}$ and M is any R-module, then*

$$\begin{aligned} M_{\mathcal{L}} =& \{(s,m) \in S \times M \mid as = 0 \text{ in } R \text{ implies } tam = 0 \\ & \text{for some } t \in S\}/\sim \end{aligned}$$

where $\sim$ is the equivalence relation given by $(s_1, m_1) \sim (s_2, m_2)$ if there exist $r_1, r_2 \in R$ such that $r_1 s_1 = r_2 s_2 \in S$ and $r_1 m_1 = r_2 m_2$.

And define addition and scalar multiplication by

$$\begin{aligned} [(s_1, m_1)] + [(s_2, m_2)] &= [(s_1 s_2, s_1 m_2 + s_2 m_1)] \\ r[(s_1, m_1)] &= [(s_1, r m_1)] \end{aligned}$$

for any $r \in R$, $s_1, s_2 \in S$ and $m_1, m_2 \in M$. Then $M_{\mathcal{L}}$ is an R-module.

Since $\mathcal{L}(C(\mathcal{P}))$ is a principal filter, as a special case of Proposition 7, *$M_{\mathcal{L}(C(\mathcal{P}))} = \{(s,m) \in C(\mathcal{P}) \times M \mid as = 0$ in R implies $tam = 0$ for some $t \in C(\mathcal{P})\}/\sim$. Hence, we may describe the modules of quotients in a rather explicit way in following theorem. This can be regarded as a generalization form of theorem in* [5].

THEOREM 7. *Let $\mathcal{P}$ be a set of prime ideals of R and M be any R-module. Then*

$$\bigcap_{P \in \mathcal{P}} M_{(P)} \cong M_{\mathcal{L}(C(\mathcal{P}))}.$$

PROOF. Let $z = m/s \in \bigcap_{P \in \mathcal{P}} M_{(P)}$. Then $m/s \in M_{(P)}$ for all $P \in \mathcal{P}$ and $s \in C(P)$. Define a map

$$\alpha : \bigcap_{P \in \mathcal{P}} M_{(P)} \longrightarrow M_{\mathcal{L}(C(\mathcal{P}))},$$

by $\alpha(z) = [(s,m)]$, where $z \in \bigcap_{P \in \mathcal{P}} M_{(P)}$. Suppose that $m_1/s_1 = m_2/m_2$ for any $m_1, m_2 \in M$, $s_1, s_2 \in C(P)$. Take $r_1 = s_2$ and $r_2 = s_1$. Then $(s_1, m_1) \sim (s_2, m_2)$, which implies $\alpha(m_1/s_1) = \alpha(m_2/s_2)$. So, α is well-defined. By the definition of scalar multiplication, α is an R-homomorphism.

Conversely, we define a map

$$\beta : M_{\mathcal{L}(C(\mathcal{P}))} \longrightarrow \bigcap_{P \in \mathcal{P}} M_{(P)}$$

by $\beta[(s,m)] = m/s$ where $[(s,m)] \in M_{\mathcal{L}(C(\mathcal{P}))}$. It is clear that β is well-defined and easily check that β is an R-homomorphism.
Moreover, $\alpha \circ \beta$ is the identity map on $M_{\mathcal{L}(C(\mathcal{P}))}$ and $\beta \circ \alpha$ is the identity map on $\bigcap_{P \in \mathcal{P}} M_{(P)}$. □

By a *torsion theory* τ we mean a pair of classes $\mathcal{T}$ and $\mathcal{F}$ of modules that satisfy the following axioms:

(1) $\mathcal{T} \cap \mathcal{F} = \{0\}$.
(2) If $M \in \mathcal{T}$ and $f : M \longrightarrow N$ is an epimorphism, then $N \in \mathcal{T}$.
(3) If $M \in \mathcal{T}$ and $N \subseteq M$, then $N \in \mathcal{T}$.
(4) If $T_a \in \mathcal{T}(a \in A :$ submodule of $M)$, then $\bigoplus_{a \in A} T_a \in \mathcal{T}$.
(5) If $0 \longrightarrow T_1 \longrightarrow T \longrightarrow T_2 \longrightarrow 0$ is exact and T_1, $T_2 \in \mathcal{T}$, then $T \in \mathcal{T}$.
(6) $F \in \mathcal{F}$ if and only if $Hom\ (T, F) = 0$ for all $T \in \mathcal{T}$.

(In many places in the literature, τ is called a *hereditary torsion theory* if it satisfies this definition). The class $\mathcal{T}$ is called the *τ-torsion class*, and the class $\mathcal{F}$ is called the *τ-torsion free class.* It is clear that τ determines a unique Gabriel filter. (cf. [2] or [8])

It is well known that there are one to one correspondences among *Idempotent kernel functors, Gabriel filters* and *Torsion theories.* Now we consider a ring of quotients which is determined by prime ideals.

For any R-module M, the module of quotients of M with respect to a torsion theory τ, denoted by $Q_\tau(M)$, is a faithfully τ-injective module containing $M/\tau(M)$ as a submodule unique up to isomorphism. Actually we can calculate

$$Q_\tau(M) = \{m \in E(M) \mid Im \subset M \quad \text{for some } I \in \mathcal{L}\}$$

where $E(M)$ is the injective hull of M. (cf. [2] or [7])

For the given Gabriel filter $\mathcal{L}(C(\mathcal{P}))$, we can define a torsion theory τ on $R - mod$. The corresponding torsion class $\mathcal{T}$ is given by

$$\{\ M \in R - \text{ mod } \mid (0 : m) \in \mathcal{L} \quad \text{for each } \ m \in M\ \}$$

where $(0 : m) = \{\ r \in R \mid rm = 0\ \}$.

PROPOSITION 8 [5, Theorem]. *Let $\mathcal{P}$ be a set of prime ideals of R and τ be the torsion theory determined by $\mathcal{L}(C(\mathcal{P}))$. Then for any τ-torsion free R-module M,*

$$Q_\tau(M) \cong \bigcap_{P\in\mathcal{P}} M_{(P)}.$$

COROLLARY 9. *Let $\mathcal{P}$ be a set of prime ideals of R and τ be the torsion theory determined by $\mathcal{L}(C(\mathcal{P}))$. Then for any τ-torsion free R-module M,*

$$M_{\mathcal{L}(C(\mathcal{P}))} \cong Q_\tau(M),$$

i.e., $Q_\tau(M) = \{(s,m)\in C(\mathcal{P})\times M \mid as = 0 \text{ in } R \text{ implies } tam = 0$ *for some* $t \in C(\mathcal{P})\}/\sim$.

References

[1] M. Beattie and M. Orzech, *Prime ideals and Finiteness conditions for Gabriel Topologies over Commutative Rings*, Rocky Mountain J. Math. **22** (1992), 423-440.

[2] J. Golan, *Torsion Theories*, Pitman Monographs and Surveys in Pure and Applied Mathematics **29** (1986), John Wiley and Sons Inc., New York.

[3] O. Goldman, *Rings and Modules of Quotients*, J. Algebra **13** (1969), 10-27.

[4] J. Han and H. Lee, *Torsion theory and Local Cohomology*, Comm. Korean Math. Soc. **4** (1989), 279-289.

[5] H. Lee, *Modules of Quotients over Commutative Rings*, Comm. Korean Math. Soc. **9** (1994), 797-801.

[6] J. Rotman, *An Introduction To Homological Algebra*, Academic Press. Inc. (1979), New York.

[7] B. Stenström, *Rings and Modules of Quotients*, Lecture Notes in Mathematics **237** (1971), Springer-Verlag, Berlin.

[8] ———, *Rings of Quotients* (1975), Springer-Verlag, Berlin.

[9] F. Van Oystaeyan and A. Verschoren, *Relative Invariants of Rings*, Monographs in pure and applied Mathematics **79** (1984), Marcel Dekker Inc., New York.

[10] I. Kaplansky, *Commutative Rings* (1974), The University of Chicago Press, Chicago and London.

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