East Asian Math J. 15(1999), No. 1, pp. 147–151

APPROXIMATION THEOREM FOR HOLOMORPHIC FUNCTIONS IN FINITE AND INFINITE DIMENSIONAL COMPLEX SPACES

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1. Introduction

We will eventually prove that every $f \in \mathcal{H}(D)$ can be approximated uniformly on compacta by polynomials in the functions $f_1, f_2, \cdots, f_n$. For the present we note some interesting properties of Oka-Weil domains. For the Banach space, J. Mujica [4] extended the Oka-Weil approximation theorem by the technique of polynomially convex set. In this paper, we obtain some properties of a sequence of polynomials which converges to a function uniformly on a polynomially convex compact subset of complex Banach spaces The technique of this study is based on J. Mujica [4].

2. Approximation theorem

A variety V in a domain $D \subset \mathbf{C}^n$ is globally presented in D if there exist functions $f_1, f_2, \cdots, f_k \in \mathcal{H}(D)$ such that $V = \{z \in D : f_1(z) = \cdots = f_k(z) = 0\}$ A variety $V \subset \mathbf{C}^n$ is regularly imbedded at a point $z_0 \in V$ if there exist holomorphic functions $f_1, f_2, \cdots, f_n$ near z_0 such that $f_1, f_2, \cdots, f_n$ form a system of local coordinates near z_0 and $V = \{z : f_1(z) = \cdots = f_k(z) = 0\}$ near z_0. If $V \subset \mathbf{C}^n$ is regularly imbedded, then all systems of defining functions $f_1, f_2, \cdots, f_k$ such that $V = \{z : f_1(z) = \cdots = f_k(z) = 0\}$ locally have Jacobian of

Received November 22, 1998. Revised January 10, 1999

1991 Mathematics Subject Classification. 32E20, 32E30.

Key words and phrases: Approximation theorem, polynomially convex set, Oka-Weil theorem, Stein manifold

The Present Studies were Supported by the Matching Fund Programs of Research Institute for Basic Sciences, Pusan National University, Korea, Project No RIBS-PNU-99-102 .

constant rank on connected components of V. An Oka-Weil manifold is the only variety which is globally presented and regularly imbedded in a polydisc. A regularly imbedded variety $V \subset \mathbf{C}^n$ has a natural complex manifold structure.

THEOREM 2.1(OKA-WEIL APPROXIMATION THEOREM [3]). *If $D \subset \mathbf{C}^n$ is a bounded analytic polyhedron determined by functions $f_1, f_2, \cdots, f_m \in \mathcal{H}(\mathbf{C}^n)$ then any function $f \in \mathcal{H}(D)$ can be approximated uniformly on compata by polynomials in the functions $f_1, f_2, \cdots, f_m$ and the variables $z_1, z_2, \cdots, z_n$.*

LEMMA 2.2. *Let V be a regularly imbedded variety in the unit open polydisc $P \subset \mathbf{C}^n$ and let f be a holomorphic function on V. Then on any open polydisc P^* with $\overline{P}^* \subset P$ there exists a function $\tilde{f} \in \mathcal{H}(P^*)$ such that $f(z) = \tilde{f}(z)$ for $z \in V \cap P^*$.*

DEFINITION 2.3. *Let M be an n-dimensional complex (analytic) manifold An open set $W \subset M$ is an Oka-Weil domain, if there exists a relatively compact open neighborhood U and $f_1, f_2, \cdots, f_m \in \mathcal{H}(M)$ such that*

(1) $W \subset \overline{W} \subset U$.

(2) $W = U \cap \{z \in M : |f_j(z)| < 1,\ 1 \leq j \leq m\}$.

(3) $F = (f_1, f_2, \cdots, f_m)$ is an injective non- singular mapping of W into the unit polydisc $P \subset \mathbf{C}^n$.

If W is an Oka-Weil domain in M, then W is Stein. If M is a Stein manifold, there exists a sequence of Oka-Weil domains $\{W_k\}$ such that $W_k \nearrow M$ and $\overline{W}_k$ is compact and $\overline{W}_k \subset W_{k+1}$ for $k = 1, 2, \cdots$.

PROPOSITION 2.4([2]). *Let M be a Stein manifold and K be a holomorphically convex compact subset of M. If U is any neighborhood of K, there is an Oka-Weil domain W, defined by global functions, such that $K \subset W \subset \overline{W} \subset U$.*

Let K be a compact subset of M and $\mathcal{A}$ be an algebra of holomorphic functions on M. The $\mathcal{A}$-convex hull of K in M is defined to be the set

$$K(\mathcal{A}, M) = \{z \in M : \mathcal{H}(z)| \leq ||f||_K \text{ for all } f \in \mathcal{A}\}.$$

We say that K is $\mathcal{A}$ - convex in M if $K = K(\mathcal{A}, M)$.

THEOREM 2.5. *Let M be a complex manifold, K be a compact subset of M and $\mathcal{A} \subset \mathcal{H}(M)$ be any subalgebra such that*

(1) $\mathcal{A}$ gives a local coordinates system at each point in M.

(2) $\mathcal{A}$ separates points in M.

(3) K is $\mathcal{A}$-convex.

Then any holomorphic function f in a neighborhood of K is approximated uniformly on K by a sequence of functions in $\mathcal{A}$.

Proof. Let U be a relatively compact open neighborhood of K. By Proposition 2.4, we have an Oka-Weil domain

$$W = \{z \in U : |f_i(z)| < 1, i = 1, 2, \cdots, m\}$$

with $f_i \in \mathcal{A}, i = 1, 2, \cdots, m$, such that $K \subset W \subset \overline{W} \subset U$. Then $\psi = (f_1, f_2, \cdots, f_m)$ maps W biholomorphically to the closed submanifold $\tilde{W} \subset P$ and $\tilde{W}$ is regularly imbedded, where P is the unit polydisc in $\mathbf{C}^m$. Since $\psi(K)$ is compact in P, there exists a polydisc P^* such that $\psi(K) \subset P^* \subset \overline{P}^* \subset P$. If $f \in \mathcal{H}(U)$ then $f \circ \psi^{-1}$ is holomorphic on $\widetilde{W}$. From Lemma 2.2, we have a function $\tilde{f} \in \mathcal{H}(P^*)$ such that $\tilde{f} = f \circ \psi^{-1}$ on $\widetilde{W} \cap P^*$. Now $\tilde{f}$ is uniformly approximated on compacta in P^* by polynomials in $z_1, z_2, \cdots, z_m$. Therefore f is approximated on K by the same polynomials in $f_1, f_2, \cdots, f_m$ and these polynomials are in $\mathcal{A}$. By repeating this argument for increasing small $U \supset K$ we have density of $\mathcal{A}$ in $\mathcal{H}(K)$.

EXAMPLE 2.6. *Let Ω be a holomorphically convex open subset of $\mathbf{C}^n$. Any compact subset of $\Omega \times \mathbf{C}^{\mathbf{N}}$ is contained in a compact set of the form $K \times L$, where K is holomorphically convex compact subset of Ω and L is a balanced convex compact subset of $\mathbf{C}^{\mathbf{N}}$. If $f \in \mathcal{H}(K \times L)$ then f depends only a finite number of coordinates and hence, by a reduction to finite dimension and an application of the finite dimensional Oka-Weil approximation theorem, f can be uniformly approximated on some neighborhood of $K \times L$ by holomorphic functions on $\Omega \times \mathbf{C}^{\mathbf{N}}$(see [1]).*

Let E and F be (complex) Banach spaces over $\mathbf{K} = \mathbf{R}$ or $\mathbf{C}$ with E finite dimensional and let $\mathcal{P}_a(E; F)$ be the vector space of all polynomials from E into F. We shall denote by $\mathcal{P}(E; F)$ the subspace of all continuous members of $\mathcal{P}_a(E; F)$.

DEFINITION 2.7. *The $\mathcal{P}(E)$-hull or polynomially convex hull of a set $A \subset E$ is defined by*

$$\hat{A}_{\mathcal{P}(E)} = \{z \in E : |P(z)| \leq \sup_A |P| \text{ for all } P \in \mathcal{P}(E)\}.$$

A compact set $K \subset E$ is said to be polynomially convex if $\hat{K}_{\mathcal{P}(E)} = K$.

EXAMPLE 2.8. *If D is a compact polydisc in $\mathbf{C}^n$, I is a finite set, and $P_i \in \mathcal{P}(\mathbf{C}^n)$ for each $i \in I$, then the compact set $\{z \in D : |P_i(z)| \leq 1$ for each $i \in I\}$ is polynomially convex. These polynomially convex compact sets are called compact polynomial polyhedra*

DEFINITION 2.9. *Let U be an open subset of E. The set U is said to be polynomially convex if $\hat{K}_{\mathcal{P}(E)} \cap U$ is compact for each compact set $K \subset U$. U is said to be strongly polynomially convex if $\hat{K}_{\mathcal{P}(E)} \subset U$ for each compact set $K \subset U$.*

DEFINITION 2.10. *E is said to be have the approximation property if for each compact set $K \subset E$ and $\epsilon > 0$ there is a finite rank operator $T \in \mathcal{L}(E;E)$ such that $\|Tz - z\| \leq \epsilon$ for every $z \in K$, where $\mathcal{L}(E;E)$ is the vector space of all linear mappings from E into E.*

The following theorem is known as the Oka-Weil theorem on polynomial approximation.

THEOREM 2.11. *Let K be a polynomially convex compact subset of $\mathbf{C}^n$. Then for each $f \in \mathcal{H}(K;F)$ there is a sequence of polynomials $P_j \in \mathcal{P}(\mathbf{C}^n;F)$ which converges to f uniformly on K.*

LEMMA 2.12. *If K is a polynomially convex compact subset of E and if U is an open neighborhood of K, then there is a strongly polynomially convex open set V with $K \subset V \subset U$.*

THEOREM 2.13. *Let K be a polynomially convex compact subset of E with the approximation property. Then for each $f \in \mathcal{H}(K;F)$ there is a sequence of polynomials $\{P_j\} \subset \mathcal{P}_f(E;F)$ which converges to f uniformly on K, where $\mathcal{P}_f(E;F)$ is the vector space of all continuous polynomials of finite type from E into F.*

Proof. If U is a polynomially convex open subset of E containing K then $f \in \mathcal{H}(U;F)$ from Lemma 2.12. Since f is continuous, we have $\delta > 0$ such that $K + B(0;\delta) \subset U$ and $\|f(z_1) - f(z_2)\| < \epsilon$ for all $z_1 \in K, z_2 \in B(z_1;\delta)$ and given $\epsilon > 0$. By assumption, there is a finite rank operator $T \in \mathcal{L}(E;E)$ such that $\|Tz_1 - z_1\| < \delta$ for every $z_1 \in K$. For every $z_1 \in K, T(K) \subset U$ and $\|f \circ T(z_1) - f(z_1)\| < \epsilon$. Since $U \cap T(E)$ is a polynomially convex open set in $T(E)$, which is finite, we have $P_j \in \mathcal{P}(T(E);F)$ such that $\|P_j(z_2) - f(z_2)\| \leq \epsilon$ for every $z_2 \in T(K)$. Since the dimension of $T(E)$ is finite, P_j is of finite type with $P_j = \sum_j c_j \varphi_j^{m_j}$ where $c_j \in F$ and $\varphi_j \in \mathcal{L}(T(E);\mathbf{K})$. Therefore, we have

$$P_j \circ T = \sum_j c_j (\varphi_j \circ T)^{m_j} \in \mathcal{P}_f(E;F).$$

Hence, we have

$$\begin{aligned}\|P_j \circ T(z) - f(z)\| &\leq \|P_j \circ T(z) - f \circ T(z)\| + \|f \circ T(z) - f(z)\| \\ &\leq 2\epsilon\end{aligned}$$

for every $z \in K$. This completes the proof.

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