East Asian Math J 15(1999), No. 1, pp. 121–133

# MULTIGRID FOR THE GALERKIN LEAST SQUARES METHOD IN PLANAR LINEAR ELASTICITY WITH P1P0 FINITE ELEMENT

JAECHIL YOO

## 1. Introduction

Let $\Omega$ be a bounded convex polygonal domain in $R^2$ and $\partial\Omega$ be the boundary of $\Omega$. The pure displacement boundary value problem for planar linear elasticity is given in the form

$$
\begin{aligned}
(1) \qquad & 2\mu\{\nabla \cdot \varepsilon(u) + \frac{\nu}{1-2\nu}\nabla\nabla\cdot u\} + f = 0 \quad \text{in } \Omega, \\
& u = 0 \quad \text{on } \partial\Omega.
\end{aligned}
$$

Here $u = (u_1, u_2)$ denotes the displacement, $f = (f_1, f_2)$ is the body force, $\nu$ is Poisson's ratio and $\mu$ is the shear modulus given by $\mu = E/\{2(1+\nu)\}$ where $E$ is the Young's modulus.
We restrict Poisson's ratio to $0 \le \nu < 1/2$ where the upper limit corresponds to an incompressible material.

Throughout this paper, we use a positive constant $C$ independent of $\nu$, mesh parameter $h_k$ and grid level $k$ which may vary from occurrence to occurrence even in the proof of the same theorem.

We define various standard differential operators as follows, see [3].

$$
\nabla \cdot v = \frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y},
$$

$$
\nabla \cdot \tau = \begin{pmatrix} \partial\tau_{11}/\partial x + \partial\tau_{12}/\partial y \\ \partial\tau_{21}/\partial x + \partial\tau_{22}/\partial y \end{pmatrix}, \quad \nabla v = \begin{pmatrix} \partial v_1/\partial x & \partial v_1/\partial y \\ \partial v_2/\partial x & \partial v_2/\partial y \end{pmatrix},
$$

Received January 27, 1999

1991 Mathematics Subject Classification 65N30, 65N55

Key words and phrases: Multigrid, Galerkin least squares method, linear elasticity, finite element

$$\tau : \eta = \sum_{i=1}^{2}\sum_{j=1}^{2}\tau_{ij}\eta_{ij}, \quad \text{and } \varepsilon(v) = \frac{1}{2}\left[\nabla v + (\nabla v)^t\right].$$

Let $H^m(\Omega)$ denote the usual Sobolev space of functions with $L^2(\Omega)$ derivatives up to order $m$. $H^m(\Omega)$ is equipped with the norm

$$\|v\|_{H^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha|\le m} |\partial^\alpha v|^2 \; dxdy\right)^{\frac{1}{2}}.$$

We use the following convention for the Sobolev seminorms:

$$|v|_{H^m(\Omega)} := \left(\int_\Omega \sum_{|\alpha|= m} |\partial^\alpha v|^2 \; dxdy\right)^{\frac{1}{2}}.$$

Let $H_0^m(\Omega) = \{v \in H^m(\Omega) : v|_{\partial\Omega} = 0\}$.

It is well known that for $f \in L^2(\Omega)$, equation (1) has a unique solution $u \in H^2(\Omega) \cap H_0^1(\Omega)$, see [5].

There is a great deal of literature dealing with approximation schemes for the equations of linear elasticity. To avoid the locking phenomenon in linear elasticity problems, there are several different approaches: nonconforming finite element methods, the methods of reduced/selected integration, first order least squares methods, and Galerkin least squares methods. For all of these approaches, mixed finite element methods involving a pair of finite element spaces are commonly used and we have to solve large linear systems arising from the finite element discretizations. With the usual mixed finite element methods, the system is indefinite and hence the problem poses difficulties.

In recent years, modern iterative methods such as multigrid and domain decomposition methods have been applied to mixed finite element methods. Among those iterative methods, the multigrid method has been one of the most popular and fastest methods. So we study the multigrid method to solve the large sparse linear systems derived from the Galerkin least squares method for the pure displacement boundary value problem.

It is well-known that one way of driving stabilized mixed finite element methods is to combine the classical Galerkin formulation with least-squares forms of the differential equations. (See [4] and references therein). An advantage of this method is that the class of finite element spaces that can be used is considerablely enlarged, hence the methods are easily incorporated into existing finite element codes. In this paper, we present a scheme of W-cycle multigrid method to solve the linear system arising from $P1P0$ conforming finite element method for the mixed formulation of the pure displacement boundary value problem as in [2], [6] and [7]. In [7], Yoo proved the convergence of W-cycle multigrid methods with $PiPj(1 \leq i, j)$ finite element.

This paper is organized as follows. We explain the conforming finite element method in section 2. We discuss the W-cycle multigrid algorithm in section 3 and prove the convergence in section 4

## 2. The Finite Element Method

For simplicity, we assume that $2\mu = 1$. Let $p = -\frac{1}{\epsilon}\nabla \cdot u$, where $\epsilon = (1-2\nu)/\nu$. Then (1) is equivalent to

$$\begin{aligned} & -\nabla \cdot \varepsilon(u) + \nabla p = f \quad \text{in } \Omega, \\ & \epsilon p + \nabla \cdot u = 0 \quad \text{in } \Omega, \\ & u = 0 \quad \text{on } \partial\Omega. \end{aligned} \tag{2}$$

Hence, we have the following weak formulation

Find $(u,p) \in H_0^1(\Omega) \times L^2(\Omega)$ such that

$$\begin{aligned} & \int_\Omega \varepsilon(u) : \varepsilon(v) \, dxdy - \int_\Omega \Big(\nabla \cdot v\Big) p \, dxdy \\ & \qquad = \int_\Omega f \cdot v \, dxdy, \quad \forall v \in H_0^1(\Omega), \\ & \epsilon \int_\Omega pq \, dxdy + \int_\Omega \Big(\nabla \cdot u\Big) q \, dxdy = 0, \qquad \forall q \in L^2(\Omega). \end{aligned} \tag{3}$$

Let $\mathcal{T}^k$ be a family of triangulations of $\Omega$, where $\mathcal{T}^{k+1}$ is obtained by connecting the midpoints of the edges of the triangles in $\mathcal{T}^k$. Let $h_T$ =diam($T$) for each $T \in \mathcal{T}^k$ and $h_k = \max_{T \in \mathcal{T}^k} h_T$, then $h_k = 2h_{k+1}$.

Now let's define the conforming finite element spaces for our multigrid method.

$$V_k := \{v \in C^0(\Omega)\ ;\ v|_T \text{ is linear for all } T \in \mathcal{T}^k \text{ and } v|_{\partial\Omega} = 0\}, \text{ and}$$
$$P_k := \{q \in L^2(\Omega)\ ;\ q|_T \text{ is a constant for all } T \in \mathcal{T}^k\}.$$

Then the discretized Galerkin least squares method for (3) is the following:

Find $(u_k, p_k) \in V_k \times P_k$ such that

$$\text{(4)} \qquad \mathcal{B}_k\Big((u_k, p_k), (v_k, q_k)\Big) = \int_\Omega f \cdot v_k \, dxdy, \ \forall (v_k, q_k) \in V_k \times P_k$$

where

$$\begin{aligned} &\mathcal{B}_k\Big((u_k, p_k), (v_k, q_k)\Big) \\ &= \int_\Omega \varepsilon(u_k) : \varepsilon(v_k) \, dxdy - \int_\Omega \Big(\nabla \cdot u_k\Big) q_k \, dxdy - \int_\Omega \Big(\nabla \cdot v_k\Big) p_k \, dxdy \\ &\quad - \beta \sum_{T \in \Gamma^k} h_T < \lceil p_k \rceil, \lceil q_k \rceil >_T - \epsilon \int_\Omega p_k q_k \, dxdy. \end{aligned}$$

Here $\Gamma^k$ stands for the collection of the element edges in the interior of $\Omega$, and $< \cdot, \cdot >$ denotes the $L^2$-inner products on $T$. $\lceil p \rceil_T$ denotes the jump in $p$ along $T$, see [4]. Note that the bilinear form $\mathcal{B}_k$ is symmetric and indefinite.
In [4], Franca and Stenberg proved the uniqueness of the solution of the conforming discretization (4) and derived the following discretization error estimate:

$$\|u - u_k\|_{H^1(\Omega)} + \|p - p_k\|_{L^2(\Omega)} \leq C h_k \|f\|_{L^2(\Omega)}$$

and

$$\|u - u_k\|_{L^2(\Omega)} \leq C {h_k}^2 \|f\|_{L^2(\Omega)}.$$

## 3. Multigird Algorithm

In this section, we discuss the W-cycle multigrid algorithm.

In order to define the fine-to-coarse operator $I_k^{k-1}$, we introduce the following mesh-dependent inner product:

$$\Big((u,p),(v,q)\Big)_k := (u,v)_{L^2(\Omega)} + h_k^2(p,q)_{L^2(\Omega)}.$$

Then $I_k^{k-1} : V_k \times P_k \to V_{k-1} \times P_{k-1}$ is defined by

$$\Big(I_k^{k-1}(u,p),(v,q)\Big)_{k-1} = \Big((u,p),(v,q)\Big)_k$$

for all $(u,p) \in V_k \times P_k$ and $(v,q) \in V_{k-1} \times P_{k-1}$.

Define $B_k : V_k \times P_k \to V_k \times P_k$ by

$$\Big(B_k(u,p),(v,q)\Big)_k = \mathcal{B}_k\Big((u,p),(v,q)\Big),$$

for all $(u,p),\ (v,q) \in V_k \times P_k$.

LEMMA 1. *The spectral radius of $B_k$ is at most $Ch_k^{-2}$.*

*Proof.* See [2].

Because of the result of Lemma 1 and indefiniteness of the system, the usual iterative methods are not appropriated to solve our linear system.

The mesh-dependent norms on $V_k \times P_k$ are defined as follows

$$|\!|\!|(u,p)|\!|\!|_{s,k} := \sqrt{\Big((B_k^2)^{s/2}(u,p),(u,p)\Big)_k} \quad \text{for all } (u,p) \in V_k \times P_k.$$

Note that $B_k$ is nonsingular and symmetric, hence ${B_k}^2$ is positive definite with respect to $(\cdot,\cdot)_k$. Therefore, this norm is well-defined for each $s \in R$. Moreover,

$$|\!|\!|(u,p)|\!|\!|_{0,k} := \sqrt{\|u\|^2_{L^2(\Omega)} + h_k^2\|p\|^2_{L^2(\Omega)}} \quad \text{for all } (u,p) \in V_k \times P_k,$$

$$\Big|\mathcal{B}_k\Big((u,p),(v,q)\Big)\Big| \le |\!|\!|(u,p)|\!|\!|_{2,k}|\!|\!|(v,q)|\!|\!|_{0,k} \quad \text{for all } (u,p),(v,q) \in V_k \times P_k,$$

and

$$|||(u,p)|||_{2,k} = \sup_{(v,q)\in V_k\times P_k\setminus\{(0,0)\}} \frac{\left|\mathcal{B}_k\Big((u,p),(v,q)\Big)\right|}{|||(v,q)|||_{0,k}}$$

for all $(u,p) \in V_k \times P_k$.

Define $P_k^{k-1} : V_k \times P_k \to V_{k-1} \times P_{k-1}$ by

$$\mathcal{B}_{k-1}\Big(P_k^{k-1}(u,p),(v,q)\Big) = \mathcal{B}_k\Big((u,p),(v,q)\Big)$$

for all $(u,p) \in V_k \times P_k$ and $(v,q) \in V_{k-1} \times P_{k-1}$.

LEMMA 2. *Given $\omega \in L^2(\Omega)$, let $(u_k,p_k) \in V_k \times P_k$ be the solution of*

$$\mathcal{B}_k\Big((u_k,p_k),(v,q)\Big) = \int_\Omega \omega \cdot v \; dxdy, \; \forall (v,q) \in V_k \times P_k$$

*and let $(u_{k-1},p_{k-1}) \in V_{k-1} \times P_{k-1}$ be the solution of*

$$\mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(v,q)\Big) = \int_\Omega \omega \cdot v \; dxdy, \; \forall (v,q) \in V_{k-1} \times P_{k-1}.$$

*Then $(u_{k-1},p_{k-1}) = P_k^{k-1}(u_k,p_k)$.*

*Proof.* Let $(u_{k-1},p_{k-1}) - P_k^{k-1}(u_k,p_k) = (\eta,\tau) \in V_{k-1} \times P_{k-1}$. Then there exists $(\zeta,\xi) \in V_{k-1} \times P_{k-1}$ such that

$$\mathcal{B}_{k-1}\Big((\zeta,\xi),(v,q)\Big) = \int_\Omega \eta \cdot v \; dxdy, \; \forall (v,q) \in V_{k-1} \times P_{k-1}.$$

Taking $(v,q) = (\eta,\tau)$, we have

$$\begin{aligned}
\|\eta\|^2_{L^2(\Omega)} &= \mathcal{B}_{k-1}\Big((\zeta,\xi),(\eta,\tau)\Big) \\
&= \mathcal{B}_{k-1}\Big((\zeta,\xi),(u_{k-1},p_{k-1})\Big) - \mathcal{B}_{k-1}\Big((\zeta,\xi),P_k^{k-1}(u_k,p_k)\Big) \\
&= \mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(\zeta,\xi)\Big) - \mathcal{B}_k\Big((u_k,p_k),(\zeta,\xi)\Big) \\
&= \int_\Omega \omega \cdot \zeta \; dxdy - \int_\Omega \omega \cdot \zeta \; dxdy \\
&= 0.
\end{aligned}$$

Since $\zeta$ is continuous, $\zeta = 0$.
Similarly, we have $\|\eta\|^2_{L^2(\Omega)} = 0$. Since $\tau$ is a piecewise constant function, $\tau = 0$.
This completes the proof.

LEMMA 3. *Given* $\omega \in L^2(\Omega)$, *let* $(u_k, p_k) \in V_k \times P_k$ *be the solution of*

$$\mathcal{B}_k\Big((u_k,p_k),(v,q)\Big) = \int_\Omega \omega q \; dxdy, \;\; \forall (v,q) \in V_k \times P_k$$

*and let* $(u_{k-1}, p_{k-1}) \in V_{k-1} \times P_{k-1}$ *be the solution of*

$$\mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(v,q)\Big) = \int_\Omega \omega q \; dxdy, \;\; \forall (v,q) \in V_{k-1} \times P_{k-1}.$$

*Then* $|\!|\!|(u_k,p_k) - (u_{k-1},p_{k-1})|\!|\!|_{0,k} \le Ch_k^2 \|\omega\|_{L^2(\Omega)}$.

*Proof.* Note that $|\!|\!|(u_k,p_k) - (u_{k-1},p_{k-1})|\!|\!|_{0,k}^2 = \|u_k - u_{k-1}\|_{L^2(\Omega)}^2 + h_k^2\|p_k - p_{k-1}\|_{L^2(\Omega)}^2$. Since $u_k - u_{k-1} \in L^2(\Omega)$, there exists a unique solution $(\zeta, \xi) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$\mathcal{B}\Big((\zeta,\xi),(v,q)\Big) = \int_\Omega (u_k - u_{k-1}) \cdot v \; dxdy \;\; \forall (v,q) \in H_0^1(\Omega) \times L^2(\Omega).$$

Let $(\zeta_k, \xi_k) \in V_k \times P_k$ be the solution of

$$\mathcal{B}_k\Big((\zeta_k,\xi_k),(v,q)\Big) = \int_\Omega (u_k - u_{k-1}) \cdot v \; dxdy, \;\; \forall (v,q) \in V_k \times P_k$$

and $(\zeta_{k-1}, \xi_{k-1}) \in V_{k-1} \times P_{k-1}$ be the solution of

$$\mathcal{B}_{k-1}\Big((\zeta_{k-1},\xi_{k-1}),(v,q)\Big) = \int_\Omega u_k - u_{k-1} \; v \; dxdy, \;\; \forall (v,q) \in V_{k-1} \times P_{k-1}$$

Taking $(v,q) = (u_k - u_{k-1}, p_k - p_{k-1})$, we have

$$\begin{aligned}
\|u_k - u_{k-1}\|_{L^2(\Omega)}^2 &= \mathcal{B}_k\Big((\zeta_k,\xi_k),(u_k,p_k) - (u_{k-1},p_{k-1})\Big) \\
&= \mathcal{B}_k\Big((u_k,p_k),(\zeta_k,\xi_k)\Big) - \mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),P_k^{k-1}(\zeta_k,\xi_k)\Big) \\
&= \mathcal{B}_k\Big((u_k,p_k),(\zeta_k,\xi_k)\Big) - \mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(\zeta_{k-1},\xi_{k-1})\Big) \\
&= \int_\Omega \omega \cdot \zeta_k \; dxdy - \int_\Omega \omega \cdot \zeta_{k-1} \; dxdy \\
&\le \|\omega\|_{L^2(\Omega)} \;\; \|\zeta_k - \zeta_{k-1}\|_{L^2(\Omega)} \\
&\le \|\omega\|_{L^2(\Omega)}\Big(\|\zeta - \zeta_k\|_{L^2(\Omega)} + \|\zeta - \zeta_{k-1}\|_{L^2(\Omega)}\Big) \\
&\le Ch_k^2 \|\omega\|_{L^2(\Omega)} \cdot \|u_k - u_{k-1}\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, we have $\|u_k - u_{k-1}\|_{L^2(\Omega)} \le Ch_k^2 \|\omega\|_{L^2(\Omega)}$.
Similarly, we have $\|p_k - p_{k-1}\|_{L^2(\Omega)} \le Ch_k \|\omega\|_{L^2(\Omega)}$. This completes the proof.

LEMMA 4. *Given $\omega \in L^2(\Omega)$, let $(u_k, p_k) \in V_k \times P_k$ be the solution of*

$$\mathcal{B}_k\Big((u_k,p_k),(v,q)\Big) = \int_\Omega \omega q \; dxdy, \; \forall (v,q) \in V_k \times P_k$$

*and let $(u_{k-1}, p_{k-1}) \in V_{k-1} \times P_{k-1}$ be the solution of*

$$\mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(v,q)\Big) = \int_\Omega \omega q \; dxdy, \; \forall (v,q) \in V_{k-1} \times P_{k-1}.$$

*Then $|\!|\!|(u_k,p_k) - (u_{k-1},p_{k-1})|\!|\!|_{0,k} \le Ch_k \|\omega\|_{L^2(\Omega)}$.*

*Proof.* Note that $|\!|\!|(u_k,p_k) - (u_{k-1},p_{k-1})|\!|\!|_{0,k}^2 = \|u_k - u_{k-1}\|_{L^2(\Omega)}^2 + h_k^2 \|p_k - p_{k-1}\|_{L^2(\Omega)}^2$. Since $u_k - u_{k-1} \in L^2(\Omega)$, there exists a unique solution $(\zeta, \xi) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying

$$\mathcal{B}\Big((\zeta,\xi),(v,q)\Big) = \int_\Omega (u_k - u_{k-1}) \cdot v \; dxdy \quad \forall (v,q) \in H_0^1(\Omega) \times L^2(\Omega).$$

Let $(\zeta_k, \xi_k) \in V_k \times P_k$ be the solution of

$$\mathcal{B}_k\Big((\zeta_k,\xi_k),(v,q)\Big) = \int_\Omega (u_k - u_{k-1}) \cdot v \; dxdy, \quad \forall (v,q) \in V_k \times P_k$$

and $(\zeta_{k-1}, \xi_{k-1}) \in V_{k-1} \times P_{k-1}$ be the solution of

$$\mathcal{B}_{k-1}\Big((\zeta_{k-1},\xi_{k-1}),(v,q)\Big) = \int_\Omega (u_k - u_{k-1}) \cdot v \; dxdy, \; \forall (v,q) \in V_{k-1} \times P_{k-1}$$

Taking $(v,q) = (u_k - u_{k-1}, p_k - p_{k-1})$, we have

$$\begin{aligned}
\|u_k - u_{k-1}\|_{L^2(\Omega)}^2 &= \mathcal{B}_k\Big((\zeta_k,\xi_k),(u_k,p_k) - (u_{k-1},p_{k-1})\Big) \\
&= \mathcal{B}_k\Big((u_k,p_k),(\zeta_k,\xi_k)\Big) - \mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}), P_k^{k-1}(\zeta_k,\xi_k)\Big) \\
&= \mathcal{B}_k\Big((u_k,p_k),(\zeta_k,\xi_k)\Big) - \mathcal{B}_{k-1}\Big((u_{k-1},p_{k-1}),(\zeta_{k-1},\xi_{k-1})\Big) \\
&= \int_\Omega \omega \xi_k \; dxdy - \int_\Omega \omega \xi_{k-1} \; dxdy \\
&\le \|\omega\|_{L^2(\Omega)} \cdot \|\xi_k - \xi_{k-1}\|_{L^2(\Omega)} \\
&\le \|\omega\|_{L^2(\Omega)} \Big(\|\xi - \xi_k\|_{L^2(\Omega)} + \|\xi - \xi_{k-1}\|_{L^2(\Omega)}\Big) \\
&\le Ch_k \|\omega\|_{L^2(\Omega)} \cdot \|u_k - u_{k-1}\|_{L^2(\Omega)}.
\end{aligned}$$

Therefore, we have

$$\|u_k - u_{k-1}\|_{L^2(\Omega)} \le Ch_k\|\omega\|_{L^2(\Omega)}. \tag{5}$$

Next, we want to estimate $\|p_k - p_{k-1}\|_{L^2(\Omega)}$. By Lemma 3.2 in [4], we have

$$\begin{aligned}
&\Big(\|u_k\|^2_{H^1(\Omega)} + (1+\epsilon)\|p_k\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}} \\
&\le C \sup_{(v,q)\in V_k\times P_k\setminus\{(0,0)\}} \frac{\Big|\mathcal{B}_k\Big((u_k,p_k),(v,q)\Big)\Big|}{\Big(\|v\|^2_{H^1(\Omega)} + (1+\epsilon)\|q\|^2_{L^2(\Omega)}\Big)^{\frac{1}{2}}} \\
&\le C \sup_{(v,q)\in V_k\times P_k\setminus\{(0,0)\}} \frac{\int_\Omega \omega q\, dxdy}{\|q\|_{L^2(\Omega)}} \\
&\le C\|\omega\|_{L^2(\Omega)}
\end{aligned}$$

Therefore, we have
$\|u_k\|_{H^1(\Omega)} + \|p_k\|_{L^2(\Omega)} \le C\|\omega\|_{L^2(\Omega)}$ and $\|u_{k-1}\|_{H^1(\Omega)} + \|p_{k-1}\|_{L^2(\Omega)} \le C\|\omega\|_{L^2(\Omega)}$. Thus

$$\|p_k - p_{k-1}\|_{L^2(\Omega)} \le \|p_k\|_{L^2(\Omega)} + \|p_{k-1}\|_{L^2(\Omega)} \le C\|\omega\|_{L^2(\Omega)}.$$

i e ,

$$h_k\|p_k - p_{k-1}\|_{L^2(\Omega)} \le Ch_k\|\omega\|_{L^2(\Omega)}. \tag{6}$$

Hence, combining (5) and (6), we obtain $\|\!|(u_k,p_k) - (u_{k-1},p_{k-1})\|\!|_{0,k} \le Ch_k\|\omega\|_{L^2(\Omega)}$.

## 4. Convergence Analysis

Now we describe the $k$-th level iteration scheme of the conforming W-cycle multigrid algorithm. The $k$-th level iteration with initial iterate $(y_0, z_0)$ yields $CMG\Big(k,(y_0,z_0),(w,r)\Big)$ as a conforming approximate solution to the following problem.

Find $(y,z) \in V_k \times P_k$ such that

$$B_k(y,z) = (w,r), \quad \text{where} \quad (w,r) \in V_k \times P_k.$$

For $k = 1$, $CMG\Big(1,(y_0,z_0),(w,r)\Big)$ is the solution obtained from a direct method.
In other words,

$$CMG\Big(1,(y_0,z_0),(w,r)\Big) = (B_1)^{-1}(w,r).$$

For $k > 1$, there are two steps.
*Smoothing step* : Let $(y_m, z_m) \in V_k \times \tilde{P}_k$ be defined recursively by the initial iterate $(y_0, z_0)$ and the equations

$$(y_l, z_l) = (y_{l-1}, z_{l-1}) + \frac{1}{\Lambda_k^2} B_k\Big((w,r) - B_k(y_{l-1}, z_{l-1})\Big), \quad 1 \le l \le m,$$

where $\Lambda_k := Ch_k^{-2}$ is greater than or equal to the spectral radius of $B_k$, and $m$ is the number of smoothings.
*Correction step* : The coarser-grid correction in $V_k \times \tilde{P}_k$ is obtained by applying the $(k-1)$-th level conforming iteration. More precisely,

$$(v_0, q_0) = (0,0) \quad \text{and}$$
$$(v_i, q_i) = CMG\Big(k-1,(v_0,q_0),(\bar{w},\bar{r})\Big), i = 1,2$$

where $(\bar{w},\bar{r}) \in V_{k-1} \times \tilde{P}_{k-1}$ is defined by $(\bar{w},\bar{r}) := I_k^{k-1}\Big((w,r) - B_k(y_m,z_m)\Big)$.
Then $CMG\Big(k,(y_0,z_0),(w,r)\Big) = (y_m,z_m) + I_{k-1}^k(v_2,q_2)$.

Now we discuss the convergence of the two-grid algorithm where the residual equation is solved exactly on the coarser grid. Let the final output of the two-grid algorithm be

$$(y^*, z^*) := (y_m, z_m) + (v^*, q^*)$$

where $(v^*, q^*) = (B_{k-1}^*)^{-1} I_k^{k-1} B_k(y - y_m, z - z_m)$.

LEMMA 5. $(v^*, q^*) = P_k^{k-1}(y - y_m, z - z_m)$.

*Proof.* See [2].

Let the $k$-th level relaxation operator $R_k$ be defined by

$$R_k := I - \frac{1}{\Lambda_k^2}(B_k)^2.$$

Then we have

$$(y - y_m, z - z_m) = R_k^m(y - y_0, z - z_0), \text{ and by Lemma 5,}$$
$$(y - y^*, z - z^*) = (I - P_k^{k-1})R_k^m(y - y_0, z - z_0).$$

LEMMA 6. **Smoothing Step** *There exists a constant $C$, independent of $h_k$ and $m$, such that*

$$|||R_k^m(u,p)|||_{2,k} \le C h_k^{-2}\frac{1}{\sqrt{m}}|||(u,p)|||_{0,k}, \quad \forall (u,p) \in V_k \times P_k.$$

*Proof.* See [2].

LEMMA 7. **Approximation Step** *There exists a constant $C$, independent of $h_k$ and $m$, such that*

$$|||(I - P_k^{k-1})(u,p)|||_{0,k} \le C h_k^2 |||(u,p)|||_{2,k}, \quad \forall (u,p) \in V_k \times P_k.$$

*Proof.* Let $(\eta, \tau) = P_k^{k-1}(u,p)$ for any $(u,p) \in V_k \times P_k$. Then $(I - P_k^{k-1})(u,p) = (u-\eta, p-\tau)$ and $|||(u-\eta, p-\tau)|||_{0,k}^2 = \|u-\eta\|_{L^2(\Omega)}^2 + h_k^2\|p-\tau\|_{L^2(\Omega)}^2$.
First, we will estimate $\|p-\tau\|_{L^2(\Omega)}$ by a duality argument. Let $(\varphi_k, \psi_k) \in V_k \times P_k$ be the solution of

$$\mathcal{B}_k\Big((\varphi_k, \psi_k), (v,q)\Big) = \int_\Omega (p-\tau)q \; dxdy, \; \forall (v,q) \in V_k \times P_k$$

and $(\varphi_{k-1}, \psi_{k-1}) \in V_{k-1} \times P_{k-1}$ be the solution of

$$\mathcal{B}_{k-1}\Big((\varphi_{k-1}, \psi_{k-1}), (v,q)\Big) = \int_\Omega (p-\tau)q \; dxdy, \; \forall (v,q) \in V_{k-1} \times P_{k-1}.$$

Then

$$\begin{aligned}\|p-\tau\|^2_{L^2(\Omega)} &= \mathcal{B}_k\Big((\varphi_k,\psi_k),(u,p)\Big) - \mathcal{B}_{k-1}\Big((\varphi_{k-1},\psi_{k-1}),(\eta,\tau)\Big)\\ &= \mathcal{B}_k\Big((\varphi_k,\psi_k),(u,p)\Big) - \mathcal{B}_{k-1}\Big((\varphi_{k-1},\psi_{k-1}),P_k^{k-1}(u,p)\Big)\\ &= \mathcal{B}_k\Big((\varphi_k,\psi_k),(u,p)\Big) - \mathcal{B}_k\Big((\varphi_{k-1},\psi_{k-1}),(u,p)\Big)\\ &= \mathcal{B}_k\Big((\varphi_k,\psi_k)-(\varphi_{k-1},\psi_{k-1}),(u,p)\Big)\\ &\le |||(\varphi_k,\psi_k)-(\varphi_{k-1},\psi_{k-1})|||_{0,k}|||(u,p)|||_{2,k}\\ &\le Ch_k\|p-\tau\|_{L^2(\Omega)}|||(u,p)|||_{2,k} \quad \text{by Lemma 4.}\end{aligned}$$

Therefore,

$$\|p-\tau\|_{L^2(\Omega)} \le Ch_k|||(u,p)|||_{2,k}. \tag{7}$$

Next, we want to estimate $\|u-\eta\|_{L^2(\Omega)}$. Let $(\zeta_k,\xi_k)\in V_k\times P_k$ be the solution of

$$\mathcal{B}_k\Big((\zeta_k,\xi_k),(v,q)\Big) = \int_\Omega (u-\eta)\cdot v\,dxdy, \quad \forall (v,q)\in V_k\times P_k \tag{8}$$

and $(\zeta_{k-1},\xi_{k-1})\in V_{k-1}\times P_{k-1}$ be the solution of

$$\begin{aligned}&\mathcal{B}_{k-1}\Big((\zeta_{k-1},\xi_{k-1}),(v,q)\Big)\\ &\qquad = \int_\Omega (u-\eta)\cdot v\,dxdy, \quad \forall (v,q)\in V_{k-1}\times P_{k-1}.\end{aligned} \tag{9}$$

Using (8) and (9), we hwve

$$\begin{aligned}\|u-\eta\|^2_{L^2(\Omega)} &= \mathcal{B}_k\Big((\zeta_k,\xi_k),(u,p)\Big) - \mathcal{B}_{k-1}\Big((\zeta_{k-1},\xi_{k-1}),(\eta,\tau)\Big)\\ &= \mathcal{B}_k\Big((\zeta_k,\xi_k),(u,p)\Big) - \mathcal{B}_{k-1}\Big((\zeta_{k-1},\xi_{k-1}),(u,p)\Big)\\ &\le |||(\zeta_k,\xi_k)-(\zeta_{k-1},\xi_{k-1})|||_{0,k}|||(u,p)|||_{2,k}\\ &\le Ch_k^2\|u-\eta\|_{L^2(\Omega)}|||(u,p)|||_{2,k} \text{by Lemma 3.}\end{aligned}$$

Therefore, we have

$$\|u-\eta\|_{L^2(\Omega)} \le Ch_k^2|||(u,p)|||_{2,k} \tag{10}$$

Hence, combining (7) and (10), we have

$$|||(I-P_k^{k-1})(u,p)|||_{0,k} \le Ch_k^2|||(u,p)|||_{2,k}.$$

THEOREM 1. **Convergence of the Two-Grid Algorithm** *There exists a constant $C$, independent of the number of levels $k$ and the number of smoothing steps $m$, such that*

$$|||(y - y^*, z - z^*)|||_{0,k} \leq \frac{C}{\sqrt{m}} |||(y - y_0, z - z_0)|||_{0,k}.$$

*Proof.* See Lemma 6 and Lemma 7.

THEOREM 2. **Convergence of the $k$-th Level Algorithm** *There exists a constant $C$, independent of the number of levels $k$ and the number of smoothing steps $m$, such that*

$$|||(y, z) - CMG(k, (y_0, z_0), (w, r))|||_{0,k} \leq \frac{C}{\sqrt{m}} |||(y - y_0, z - z_0)|||_{0,k}.$$

*Proof.* See [1].

## References


[1] R. E. Bank and T. Dupont, *An optimal order process for solving finite element equations*, Math. Comp., **36** (1981), 35-51

[2] S. C. Brenner, *A nonconforming mixed multigrid method for the pure displacement problem in planar linear elasticity*, SIAM J. Numer. Anal., **30** (1993), 116-135

[3] P. Cialet, *The finite element method for elliptic problems*, North-Holland, Amsterdam, 1978

[4] L. P. Franca and R. Stenberg, *Error analysis of some Galerkin least squares methods for the elasticity equations*, SIAM J. Numer. Anal. **28** (1991), 1680-1697

[5] P.Grisvard, *Singularités en elasticité*, Arch. Rational Mech. Anal. **107** (1989), 157-180

[6] C.-O. Lee, *Multigrid methods for the pure traction problem of linear elasticity mixed formulation*, SIAM J. Numer. Anal. **35** (1998), 121-145

[7] J. Yoo, *Multigrid method for the Galerkin least squares method in linear elasticity. The pure displacement problem*, Proceedings of the 9th Copper Mountain Conference on iterative methods, Cray Research, Inc. 1996



Department of Mathematics
Dongeui University
Pusan 614-714, Korea
*E-mail* : yoo@hyomin.dongeui.ac.kr