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THE RADIUS OF CONVEXITY FOR THE CLASS $K^{(2)}$

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1. Introduction

Let S denote the class of functions f of a complex variable z, analytic and univalent in the open unit disk $\Delta = \{z : |z| < 1\}$, and normalized by $f(0) = f'(0) - 1 = 0$ and hence with the Taylor expansion

$$f(z) = z + a_2 z^2 + \cdots + a_n z^n + \cdots, \quad z \in \Delta.$$

Let K denote the subclass of S consisting of functions f for which $f(\Delta)$ is a convex set. Furthermore, let $S^{(2)}$ denote the class of odd functions in S, i e., the functions with the expansion

$$g(z) = z + c_3 z^3 + c_5 z^5 + \cdots + c_{2n+1} z^{2n+1} + \cdots, \quad z \in \Delta.$$

For each function $f \in S$, the square root transform

$$g(z) = \sqrt{f(z^2)} = z + c_3 z^3 + c_5 z^5 + \cdots$$

is an odd univalent function. Conversely, it is easy to see that every odd function $g \in S$ is the square-root transform of some $f \in S$. We define $K^{(2)}$ be the class of functions which are square-root transforms of functions in K.

The one of the geometric properties for the class S is that every $f(z)$ in S is not convex. Near the origin each function $f \in S$ is close to the identity mapping. It is to be expected that f will map small circles $|z| = \rho$ onto curves which bound convex domains.

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THEOREM 1.1. *[1] For every positive number $\rho \leq 2-\sqrt{3}$, each function $f \in S$ maps the disc $|z| < \rho$ onto a convex domain. This is false for every $\rho > 2-\sqrt{3}$.*

This number $\rho = 2-\sqrt{3} = 0.267\ldots$ is called the radius of convexity for the class S. Let $h(z) = z(1-z)^{-1} \in K$. Then we have $\sqrt{h(z^2)} \notin K$, i.e., $K^{(2)}$ is not the subclass of K. Thus we would find the radius of convexity for the class $K^{(2)}$.

2. Preliminaries

THEOREM 2.1. *([1], Growth and Distortion theorem) If $f \in S$ and $|z| = r < 1$ then*

$$\frac{r}{(1+r)^2} \leq |f(z)| \leq \frac{r}{(1-r)^2}$$

and

$$\frac{1-r}{(1+r)^3} \leq |f'(z)| \leq \frac{1+r}{(1-r)^3}.$$

For each $z \in \Delta$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

THEOREM 2.2. *[1] For each $f \in S$,*

$$\frac{1-r}{1+r} \leq \left|\frac{zf'(z)}{f(z)}\right| \leq \frac{1+r}{1-r}, \quad |z| = r < 1.$$

For each $z \in \Delta, z \neq 0$, equality occurs if and only if f is a suitable rotation of the Koebe function.

THEOREM 2.3. *For odd functions $h \in S^{(2)}$*

$$\frac{r}{1+r^2} \leq |h(z)| \leq \frac{r}{1-r^2}$$

and

$$\frac{1-r^2}{(1+r^2)^2} \leq |h'(z)| \leq \frac{1+r^2}{(1-r^2)^2}, \quad |z| = r < 1.$$

Proof. Let $h(z) = \sqrt{f(z^2)}$ for some $f \in S$, then

$$\sqrt{\frac{r^2}{(1+r^2)^2}} \leq |h(z)| \leq \sqrt{\frac{r^2}{(1-r^2)^2}}.$$

Thus
$$\frac{r}{1+r^2} \le |h(z)| \le \frac{r}{1-r^2}, \quad |z| = r < 1.$$

Since
$$\frac{1-r}{1+r} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1+r}{1-r}$$

and
$$\frac{zh'(z)}{h(z)} = \frac{z^2 f'(z^2)}{f(z^2)} \ ,$$
$$\frac{1-r^2}{1+r^2} < \left|\frac{zh'(z)}{h(z)}\right| < \frac{1+r^2}{1-r^2}$$

and
$$|h'(z)| = \left|\frac{zf'(z^2)h(z)}{f(z^2)}\right|, \quad |z| = r < 1.$$

Thus $\dfrac{1-r^2}{(1+r^2)^2} \le |h'(z)| \le \dfrac{1+r^2}{(1-r^2)^2}, \quad |z| = r < 1.$

3. Main Results

LEMMA 3.1. *For each $f \in K$,*

$$\frac{1}{(1+r)^2} \le |f'(z)| \le \frac{1}{(1-r)^2}, \quad |z| = r < 1\,.$$

For each $z \in \Delta$, $z \neq 0$, equality occurs if and only if f is a suitable rotation of the function $l(z) = z(1-z)^{-1}$.

LEMMA 3.2. *For convex function $f \in K$,*

$$\frac{r}{1+r} \le |f(z)| \le \frac{r}{1-r}, \quad |z| = r < 1,$$

with equality occurring only for functions of the form

$$f(z) = \frac{z}{1-e^{i\varphi}z}, \quad 0 \le \varphi \le 2\pi\,.$$

The growth of $K^{(2)}$ would be obtained by the following theorem.

THEOREM 3.3. *For $h \in K^{(2)}$,*

$$\frac{r}{\sqrt{1+r^2}} \le |h(z)| \le \frac{r}{\sqrt{1-r^2}}, \quad |z| = r < 1\,.$$

Proof. Let $h(z) = \sqrt{f(z^2)}$ and $f \in K$. Then by Lemma 3.2,

$$|h(z)| = |\sqrt{f(z^2)}| \le \sqrt{\frac{r^2}{1-r^2}} = \frac{r}{\sqrt{1-r^2}}$$

and

$$\frac{r}{\sqrt{1+r^2}} \le |h(z)|\,, \quad |z| = r < 1.$$

If $h \in K^{(2)}$, then we have

$$\frac{r}{1+r} \le \frac{r}{\sqrt{1+r^2}} \le |h(z)| \le \frac{r}{\sqrt{1-r^2}} \le \frac{r}{1-r}, \quad |z| = r < 1$$

But $K^{(2)}$ is not the subclass of convex functions.

LEMMA 3.4. *For each $f \in K$,*

$$\frac{1}{1+r} \le \left|\frac{zf'(z)}{f(z)}\right| \le \frac{1}{1-r}\;, \quad |z| = r < 1.$$

For each $z \in \Delta, z \ne 0$, equality occurs if and only if f is a suitable rotation of the function $l(z) = z/(1-z)$.

LEMMA 3.5. *For each $f \in K$,*

$$-\frac{2r}{1+r} \le Re\left\{\frac{zf''(z)}{f'(z)}\right\} \le \frac{2r}{1-r}\;, \quad |z| = r < 1.$$

THEOREM 3.6. *For every positive number $\sigma \le \sqrt{5-\sqrt{17}}/2$, each function $h \in K^{(2)}$ maps the disk $\Delta_\sigma = \{z : |z| < \sigma\}$ onto a convex domain and $\sqrt{5-\sqrt{17}}/2 > 2 - \sqrt{3}$*

Proof. For each $f \in K$ and $h = \sqrt{f(z^2)} \in K^{(2)}$,

$$Re\left\{1 + \frac{zh''(z)}{h'(z)}\right\} = Re\left\{2 + \frac{2z^2 f''(z^2)}{f'(z^2)} - \frac{z^2 f'(z^2)}{f(z^2)}\right\}$$

and

$$Re\left\{1+\frac{zh''(z)}{h'(z)}\right\}>0, \quad |z|=r<\frac{\sqrt{5-\sqrt{17}}}{2}$$

by Lemma 3.4 and 3.5. Thus h maps such a disk $\{z : |z| < \sqrt{5-\sqrt{17}}/2\}$ onto a convex domain

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