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ON AN EVALUATION OF $_3F_2(1/2)$

JUNESANG CHOI AND ARJUN K. RATHIE

1. Introduction

The generalized hypergeometric function with p numerator and q denominator parameters is defined by

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}\alpha_1, \ldots, \alpha_p\,; \\ \beta_1, \ldots, \beta_q\,;\end{matrix}\; z\right] = {}_pF_q\left[\alpha_1, \ldots, \alpha_p; \beta_1, \ldots, \beta_q; z\right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!},$$

where $(\alpha)_n$ denotes the Pochhammer symbol (or *the shifted factorial*, since $(1)_n = n!$) defined by

$$(\alpha)_n := \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} & (n \in \mathbf{N}) \\ 0 & (n = 0), \end{cases}$$

for any complex number α, Γ the well-known Gamma function, and $\mathbf{N}$ the set of natural numbers.

The following interesting and well known definite integral has been recorded in various literature (e.g., see [2, p. 99, Entry 15.94]):

$$(1.2)\qquad \int_0^1 \frac{\log(1-x)}{x} dx = -\frac{\pi^2}{6}$$

which can be easily evaluated by using Maclaurin's expansion of $\log(1-x)$ and term-wise integration.

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The object of this note is to find the value of

$$ {}_3F_2(1,\,1,\,1\,;\,2,\,2\,;\,1/2) \tag{1.3} $$

by evaluating the integral

$$ \int_0^{1/p} \frac{\log(1-x)}{x}\,dx \tag{1.4} $$

in two ways and then setting $p = 2$. As in ${}_2F_1(1/2)$, it has not yet been found to evaluate ${}_3F_1(1/2)$ generally (see [1, pp. 45–107]). And so the evaluation of its special cases is naturally considered. Indeed, the summation formula to be proved is

$$ {}_3F_2\left(1,\,1,\,1\,;\,2,\,2\,;\,\frac{1}{2}\right) = \frac{\pi^2}{6} - (\log 2)^2. \tag{1.5} $$

2. Derivation of the Formula (1.5)

Using the Maclaurin's series expansion of $\log(1-x)$ and term-wise integration, it is not difficult to see that, for $p = 2,\,3,\,\ldots$

$$ \begin{aligned} \int_0^{1/p} \frac{\log(1-x)}{x}\,dx &= -\sum_{n=1}^{\infty} \frac{1}{n^2 p^n} \\ &= -\sum_{n=0}^{\infty} \frac{1}{(n+1)^2 p^{n+1}} \\ &= -\frac{1}{p}\sum_{n=0}^{\infty} \frac{1}{(n+1)^2 p^n}. \end{aligned} \tag{2.1} $$

Hence

$$ \int_0^{1/p} \frac{\log(1-x)}{x}\,dx = -\frac{1}{p}\,{}_3F_2\left(1,\,1,\,1\,;\,2,\,2\,;\,\frac{1}{p}\right). \tag{2.2} $$

On the other hand we separate the integral (1.2) into two parts as in the following way

$$ I := \int_0^1 \frac{\log(1-x)}{x}\,dx = I_1 + I_2, \tag{2.3} $$

where

$$I_1 = \int_0^{1/p} \frac{\log(1-x)}{x} dx,$$
$$I_2 = \int_{1/p}^{1} \frac{\log(1-x)}{x} dx.$$

Now, for I_2, performing integration by parts, we have after some simplification

$$\begin{aligned} I_2 &= \log p \log\left(1 - \frac{1}{p}\right) + \int_{1/p}^{1} \frac{\log x}{1-x} dx \\ &:= \log p \log\left(1 - \frac{1}{p}\right) + I_3. \end{aligned} \tag{2.4}$$

Further for I_3, let $x = 1 - t$ and simplifying, we get

$$I_3 = \int_0^{1-\frac{1}{p}} \frac{\log(1-x)}{x} dx. \tag{2.5}$$

Hence by (2.3), using (2.4) and (2.5), we have

$$\begin{aligned} \int_0^1 \frac{\log(1-x)}{x} dx &= \log p \log\left(1 - \frac{1}{p}\right) \\ &\quad + \int_0^{1/p} \frac{\log(1-x)}{x} dx + \int_0^{1-\frac{1}{p}} \frac{\log(1-x)}{x} dx. \end{aligned} \tag{2.6}$$

Now setting $p = 2$ in (2.2), and using (1.2), we get

$$-\frac{1}{2} {}_3F_2\left(1, 1, 1\,;\, 2, 2\,;\, \frac{1}{2}\right) = \int_0^{1/2} \frac{\log(1-x)}{x} dx \tag{2.7}$$

and setting $p = 2$ in (2.6) and using (1.2), we get

$$\frac{1}{2}\left(-\frac{\pi^2}{6} + (\log 2)^2\right) = \int_0^{1/2} \frac{\log(1-x)}{x} dx. \tag{2.8}$$

Hence our desired result (1.5) follows from (2.7) and (2.8).

References

[1] E. D. Rainville, *Special Functions*, The Macmillian Company, New York, 1960.
[2] M. R. Spiegel, *Mathematical Handbook of Formulas and Tables*, Schaum's Outline Series, McGraw-Hill Book Company, New York, 1968.

Junesang Choi
Department of Mathematics
College of Natural Sciences
Dongguk University
Kyongju 780-714, Korea
E-mail : junesang@mail.dongguk.ac.kr

Arjun K. Rathie
Department of Mathematics
Government Dungar College
M. D. S. University
Bikaner-334001
Rajasthan State, India