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SYMMETRIC BI-DERIVATIONS ON PRIME RINGS

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1. Introduction

In [6], J. Vukman has proved some results concerning symmetric bi-derivation on prime and semi-prime rings. In this short note, we obtain a few results on symmetric bi-derivations in prime rings.

2. Preliminaries

Throughout this paper all rings will be associative. Denote by R (resp., C and Z) an associative ring (resp , the extended centroid of R and the center of R). We shall write $[x,y]$ for $xy - yx$. A mapping $D(-,-) : R \times R \to R$ is said to be *symmetric* if $D(x,y) = D(y,x)$ for all $x, y \in R$. In what follows, denote by $D(-,-)$ a symmetric mapping from $R \times R$ to R without otherwise specified A mapping $d : R \to R$ is called the *trace* of $D(-,-)$ if $d(x) = D(x,x)$for all $x \in R$ It is obvious that if $D(-,-)$ is bi-additive (i.e , additive in both arguments), then the trace d of $D(-,-)$ satisfies the identity $d(x+y) = d(x) + d(y) + 2D(x,y)$ for all $x, y \in R$ If $D(-,-)$ is bi-additive and satisfies the identity $D(xy,z) = D(x,z)y + xD(y,z)$ for all $x, y, z \in R$, we say that $D(-,-)$ is a *symmetric bi-derivation*

LEMMA 2.1 [1, Lemma 3 1.1]). *Let R be a prime ring with* char$R$ $\neq 2$, *$D(-,-)$ a symmetric bi-derivation and d the trace of $D(-,-)$. If U is a non-zero ideal of R such that $ad(U) = 0$* (or, *$d(U)a = 0$*), *then $a = 0$ or $d = 0$.*

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Lemma 2.2 [1, Theorem 3.1.3]). *Let R be a prime ring with* char$R$ $\neq 2$, *$D(-,-)$ a symmetric bi-derivation and d the trace of $D(-,-)$. For a fixed element $a \in R$, we have*

(i) *if $[a, d(x)] = 0$ for all $x \in R$, then $a \in Z$ or $d = 0$.*

(ii) *if $[a, d(x)] \in Z$ for all $x \in R$ and for non-zero trace d with $d(a) \neq 0$, then $a \in Z$.*

Lemma 2.3 [3, Lemma 2]). *Let R be a prime ring and let $a, b, c \in R$. If $axb = cxa$ for all $x \in R$, then $a = 0$ or $b = c$.*

3. Main results

We begin with the following lemma.

Lemma 3.1. *Let R be a prime ring with* char$R \neq 2$ *and let d_1 and d_2 be traces of symmetric bi-derivations $D_1(-,-)$ and $D_2(-,-)$, respectively. If the identity*

$$d_1(x)d_2(y) = d_2(x)d_1(y) \tag{1}$$

holds and $d_1 \neq 0$, then there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$.

Proof. Let $x, y, z \in R$. Replacing y by $y + z$ in (1), we get

$$d_1(x)D_2(y,z) = d_2(x)D_1(y,z), \tag{2}$$

and replacing z by zy in (2) leads to the identity

$$d_1(x)zd_2(y) = d_2(x)zd_1(y). \tag{3}$$

It follows from replacing y by x in (3) that

$$d_1(x)zd_2(x) = d_2(x)zd_1(x). \tag{4}$$

Thus if $d_1(x) \neq 0$, then by (4) and [4, Corollary to Lemma 1.3.2] we have $d_2(x) = \lambda(x)d_1(x)$ for some $\lambda(x) \in C$. Hence if $d_1(x) \neq 0$ and $d_1(y) \neq 0$, then $(\lambda(y) - \lambda(x))d_1(x)zd_1(y) = 0$ by (3). Since R is prime, it follows from Lemma 2.1 that $\lambda(x) = \lambda(y)$. This shows that there exists $\lambda \in C$ such that $d_2(x) = \lambda d_1(x)$ under the condition $d_1(x) \neq 0$. On the other hand, assume that $d_1(x) = 0$. Since $d_1 \neq 0$ and R is prime, it follows from (3) that $d_2(x) = 0$ as well. Thus $d_2(x) = \lambda d_1(x)$. This completes the proof.

THEOREM 3.2. *Let R be a prime ring with* $\mathrm{char} R \neq 2$ *and let* $d_1(\neq 0)$, d_2, d_3, *and* $d_4(\neq 0)$ *be traces of symmetric bi-derivations* $D_1(-,-)$, $D_2(-,-)$, $D_3(-,-)$, *and* $D_4(-,-)$ *respectively. If the identity*

$$d_1(x)d_2(y) = d_3(x)d_4(y) \tag{5}$$

holds for all $x, y \in R$, *then there exists* $\lambda \in C$ *such that* $d_2(x) = \lambda d_4(x)$ *and* $d_3(x) = \lambda d_1(x)$.

Proof. Let $x, y, z, w \in R$. Replacing y by $y + z$ in (5), we get

$$d_1(x)D_2(y,z) = d_3(x)D_4(y,z), \tag{6}$$

and replacing z by zy in (6) and using (6) leads to the identity

$$d_1(x)zd_2(y) = d_3(x)zd_4(y). \tag{7}$$

It follows from replacing z by $zd_4(w)$ in (7) that

$$d_1(x)zd_4(w)d_2(y) = d_3(x)zd_4(w)d_4(y) = d_1(x)zd_2(w)d_4(y),$$

so that $d_1(x)z(d_4(w)d_2(y) - d_2(w)d_4(y)) = 0$. Since $d_1 \neq 0$ and R is prime, it follows that $d_4(w)d_2(y) = d_2(w)d_4(y)$. Applying Lemma 3.1, there exists $\lambda \in C$ such that $d_2(y) = \lambda d_4(y)$, which implies from (7) that $(\lambda d_1(x) - d_3(x))zd_4(y) = 0$ so that $d_3(x) = \lambda d_1(x)$. This completes the proof.

THEOREM 3.3. *Let R be a prime ring with* $\mathrm{char} R \neq 2, 3$ *and let d be the trace of a non-zero symmetric bi-derivation* $D(-,-)$ *For a fixed element a of R with* $d(a) \neq 0$, *if the identity*

$$d(x)ad(x) = 0 \tag{8}$$

holds for all $x \in R$, *then* $a \in Z$.

Proof. By linearizing (8) and using (8), we get

$$\begin{aligned} &d(x)ad(y) + 2d(x)aD(x,y) + d(y)ad(x) + 2d(y)aD(x,y) \\ &+ 2D(x,y)ad(x) + 2D(x,y)ad(y) + 4D(x,y)aD(x,y) = 0 \end{aligned} \tag{9}$$

for all $x, y \in R$. Substituting $-x$ for x in (9), we have

$$\begin{aligned} & d(x)ad(y) - 2d(x)aD(x,y) + d(y)ad(x) \\ & - 2d(y)aD(x,y) - 2D(x,y)ad(x) - 2D(x,y)ad(y) \\ (10) \quad & + 4D(x,y)aD(x,y) = 0. \end{aligned}$$

By adding (9) and (10), and using the fact that $\text{char}R \neq 2$, we obtain

$$(11) \qquad d(x)ad(y) + d(y)ad(x) + 4D(x,y)aD(x,y) = 0.$$

Now we substitute $x + y$ for x in (11) and expand it, and then we use (8), (11) and the fact that $\text{char}R \neq 2$. Then we obtain

$$(12) \quad D(x,y)ad(y) + d(y)aD(x,y) + 2d(x)aD(x,y) + 2D(x,y)ad(x) = 0.$$

Replacing y by $x+y$ in (12) and then using (8), (11), (12) and the fact that $\text{char}R \neq 3$, we get

$$(13) \qquad D(x,y)ad(x) + d(x)aD(x,y) = 0$$

Substituting yz for y in (13), and reminding that

$$D(x,y)ad(y) = -d(y)aD(x,y) \quad \text{and} \quad D(z,y)ad(y) = -d(y)aD(z,y),$$

we can write

$$(14) \qquad D(x,y)[z, ad(y)] = [x, d(y)a]D(z,y).$$

Replacing x by xw in (14) and using (14) again, we have

$$(15) \qquad D(x,y)w[z, ad(y)] = [x, d(y)a]wD(z,y).$$

Exchanging z for x in (15); then

$$(16) \qquad D(x,y)w[x, ad(y)] = [x, d(y)a]wD(x,y).$$

It follows from Lemma 2.3 that $D(x,y) = 0$ or $[x, ad(y)] = [x, d(y)a]$. In other words, R is the union of its subsets $A := \{x \in R | D(x,y) = 0 \text{ for all } y \in R\}$ and $B := \{x \in R | [x, ad(y) - d(y)a] = 0 \text{ for all } y \in R\}$. Note that A and B are additive subgroups of R. Since R can't be written as the union of A and B, it follows that $A = R$ or $B = R$ so from the hypothesis that $R = B$. This implies that $[a, d(y)] \in Z$ for all $y \in R$. By Lemma 2.2(ii), we know that $a \in Z$. This completes the proof.

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