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LOCAL EXISTENCE OF INTEGRAL SOLUTIONS FOR NONLINEAR DIFFERENTIAL EQUATIONS IN L^p SPACES

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1. Introduction

Let X be a real Banach space. Let $A : X \supset D(A) \to 2^X$ and let $u_0 \in \overline{D(A)}$. In this paper, we are concerned with the nonlinear differential equation of the form

$$(1.1)\qquad \begin{cases} \dfrac{du(t)}{dt} + Au(t) \ni G(t, u(t), L_t u), \quad 0 \le t \le T, \\ u(0) = u_0 \end{cases}$$

for $T > 0$, where $G : [0,T] \times X \times X \to X$, for every $t \in [0,T]$, $L_t : L^p(0,t;X) \to X$ and for $1 \le p < \infty$.

Shioji([7]) has shown the existence of the local solutions for the initial value problem of the form

$$\begin{cases} \dfrac{du(t)}{dt} + Au(t) \ni Gu(t), \quad 0 \le t \le T, \\ u(t) = u_0 \end{cases}$$

with continuity of $G : L^p(0,T;X) \to L^1(0,T;X)$ using Schauder's fixed point theorem in $L^p(0,T;X)$ for $1 \le p < \infty$ instead of that in $C([0,T];X)$ which many mathematicians have used.

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We will show the local existence of integral solutions for (1.1) via compactness method with Lipschitz continuity of $G : [0,T] \times X \times X \to X$ and $L_t : L^p(0,t;X) \to X$ for every $t \in [0,T]$, and $u_0 \in \overline{D(A)}$ using Schauder's fixed point theorem in $L^p(0,T;X)$. The term $G(t, u(t), L_t u)$ is a similar form in Kartsatos-Liu([5]), and in Ha-Shin-Jin([4]) when $L_t u = \int_0^t k(t,s,u(s))ds$ for every $t \in [0,T]$.

In the next section, we state some preliminaries and the third section is devoted to the local existence of solutions. In section 4, we consider an example.

2. Preliminaries

Let X be a real Banach space with its norm $\|\cdot\|$. We define a function $[\cdot,\cdot]_+ : X \times X \to \mathcal{R}$ by

$$[x,y]_+ = \lim_{t\to 0+} \frac{\|x+ty\| - \|x\|}{t}$$

for every $[x,y] \in X \times X$.

Let $A : X \supset D(A) \to 2^X$ be an operator, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. A function $u : [0,T] \to X$ is called an integral solution of the initial value problem

$$\text{(2.1)} \qquad \begin{cases} \dfrac{du(t)}{dt} + Au(t) \ni f(t), \quad 0 \le t \le T, \\ u(0) = u_0 \end{cases}$$

if u is continuous on $[0,T]$, $u(0) = u_0$, $u(t) \in \overline{D(A)}$ for every $t \in [0,T]$ and

$$\|u(t) - x\| \le \|u(s) - x\| + \int_s^t [u(\tau) - x, f(\tau) - y]_+ d\tau$$

for every $[x,y] \in A$ and $0 \le s \le t \le T$.

The reader is referred to Barbu([2]) and Lakshmikantham-Leela([6]) for accretive operators and nonlinear differential equations in Banach spaces.

We need the following results for the local existence of integral solutions of (1.1) in the next section.

PROPOSITION 1. (Bénilan [3]) *Let $A : X \supset D(A) \to 2^X$ be an m-accretive operator, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. Then (2.1) has a unique integral solution. Let u and v be the integral solutions of (2.1) corresponding to (f, u_0), $(g, v_0) \in L^1(0,T;X) \times \overline{D(A)}$, respectively. Then*

$$\|u(t) - v(t)\| \leq \|u(s) - v(s)\| + \int_s^t \|f(\tau) - g(\tau)\| d\tau$$

for $0 \leq s \leq t \leq T$.

PROPOSITION 2. (Vrabie [8]) *Let $A : X \supset D(A) \to 2^X$ be an m-accretive operator, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. Let u be the unique integral solution of (2.1) and $\{S(t) \mid t \geq 0\}$ be the nonlinear semigroup on $\overline{D(A)}$ generated by $-A$. Then*

$$\|u(t+s) - u(t)\| \leq \int_0^s \|f(\tau)\| d\tau + \|S(s)u_0 - u_0\| + \int_0^{T-s} \|f(\tau + s) - f(\tau)\| d\tau$$

for t, $s > 0$ with $t + s \leq T$.

PROPOSITION 3. (Baras [1]) *Let $A : X \supset D(A) \to 2^X$ be an m-accretive operator, $f \in L^1(0,T;X)$ and $u_0 \in \overline{D(A)}$. Assume that the nonlinear semigroup $\{S(t) \mid t \geq 0\}$ on $\overline{D(A)}$ generated by $-A$ is compact. Then for every bounded subset B of $L^1(0,T;X)$, the set $\{u^f \in C([0,T];X) \mid f \in B\}$ of integral solutions of (2.1) is relatively compact in $L^p(0,T,X)$ for $1 \leq p < \infty$*

PROPOSITION 4. (Shioji [7]) *Let $A : X \supset D(A) \to 2^X$ be an m-accretive operator and $u_0 \in \overline{D(A)}$. Let B be a bounded subset of $L^1(0,T;X)$ such that*

$$\lim_{h \to 0+} \int_0^{T-h} \|f(t+h) - f(t)\| dt = 0, \quad \lim_{h \to 0+} \int_0^h \|f(t)\| dt = 0$$

uniformly for $f \in B$. Assume that the resolvent $J_\lambda = (I + \lambda A)^{-1}$ of A is compact for every $\lambda > 0$ Then the set $\{u^f \in C([0,T];X) \mid f \in B\}$ of integral solutions of (2.1) is relatively compact in $L^p(0,T;X)$ for $1 \leq p < \infty$.

3. Local existence theorem

DEFINITION 1. A function $u(t) : [0,T] \to X$ is called an integral solution of (1.1) if $u(t) : [0,T] \to X$ is an integral solution of (2.1) for $f(t) = G(t, u(t), L_t u)$.

We consider the following conditions.

(A) $A : X \supset D(A) \to 2^X$ is an m-accretive operator.

(L) For every $t \in [0,T]$, $L_t : L^p(0,T;X) \to X$ for $1 \leq p < \infty$ satisfies

$$\text{(L1)} \qquad ||L_t u - L_t v|| \leq a(t)||u - v||_{t,p},$$
$$\text{(L2)} \qquad ||L_t u - L_s u|| \leq r_1(||u||_{T,p})|t - s|$$

for every $t, s \in [0,T]$ and $u, v \in L^p(0,T;X)$, where $a : [0,T] \to [0,\infty)$ is continuous, $r_1 : [0,\infty) \to [0,\infty)$ is nondecreasing and $||u||_{t,p} = (\int_0^t ||u(\tau)||^p d\tau)^{\frac{1}{p}}$ for every $u \in L^p(0,T,X)$ with $t \in [0,T]$ and $1 \leq p < \infty$.

(G) $G : [0,T] \times X \times X \to X$ satisfies

(G1)
$$||G(t,w,x) - G(t,y,z)|| \leq b(t)(||w - y|| + ||x - z||),$$
(G2)
$$||G(t,x,y) - G(s,x,y)|| \leq r_2(||x||, ||y||)|t - s|$$

for every $t, s \in [0,T]$ and $w, x, y, z \in X$, where $b : [0,T] \to [0,\infty)$ is continuous and $r_2 : [0,\infty) \times [0,\infty) \to [0,\infty)$ is nondecreasing for both variables.

Assume that (L) and (G) are satisfied. We put $a_T = \max_{t\in[0,T]} a(t)$ and $b_T = \max_{t\in[0,T]} b(t)$. Let $u \in L^p(0,T;X)$. From (L), for every $t \in [0,T]$,

$$\begin{aligned} ||L_t u|| &\leq ||L_t u - L_t 0|| + ||L_t 0 - L_0 0|| + ||L_0 0|| \\ &\leq a(t)||u||_{t,p} + r_1(0)t + ||L_0 0|| \\ &\leq a_T||u||_{T,p} + r_1(0)T + ||L_0 0|| \\ &= a_T||u||_{T,p} + c_T, \end{aligned}$$

where $c_T = r_1(0)T + ||L_0 0||$, and thus from (G), for every $t \in [0,T]$,

$$\begin{aligned}
&||G(t,u(t),L_t u)|| \\
&\quad \le ||G(t,u(t),L_t u) - G(t,0,0)|| + ||G(t,0,0) - G(0,0,0)|| \\
&\qquad + ||G(0,0,0)|| \\
&\quad \le b(t)(||u(t)|| + ||L_t u||) + r_2(0,0)t + ||G(0,0,0)|| \\
&\quad \le b_T||u(t)|| + b_T(a_T||u||_{T,p} + c_T) + r_2(0,0)T + ||G(0,0,0)||.
\end{aligned}$$

Let $\frac{1}{p} + \frac{1}{q} = 1$. Hence for every $u \in L^p(0,T;X)$,

$$\begin{aligned}
&\int_0^T ||G(t,u(t),L_t u)||dt \\
&\quad \le b_T \int_0^T ||u(t)||dt + \{b_T(a_T||u||_{T,p} + c_T) \\
&\qquad + r_2(0,0)T + ||G(0,0,0)||\}T \\
&\quad \le b_T(T^{\frac{1}{q}} + a_T T)||u||_{T,p} + \{b_T c_T + r_2(0,0)T \\
&\qquad + ||G(0,0,0)||\}T.
\end{aligned} \tag{3.1}$$

First of all, we consider two local existence theorems under the assumption of compactness of the nonlinear semigroup.

THEOREM 1. *Assume that* (A), (L) *and* (G) *are satisfied and the nonlinear semigroup* $\{S(t) \mid t \ge 0\}$ *on* $\overline{D(A)}$ *generated by* $-A$ *is compact. Let* $u(0) \in \overline{D(A)}$. *Then* (1.1) *has a local integral solution*

Proof. Choose $0 < T_0 \le T$ and $\varepsilon > 0$ such that $T_0^{\frac{1}{p}} M_{T_0} \le \varepsilon$ and

$$\int_0^{T_0} ||G(t,u(t),L_t u)||dt \le M_{T_0} \tag{3.2}$$

for every $u \in L^p(0,T_0;X)$ with $||u - S(\cdot)u_0||_{T_0,p} \le \varepsilon$, where

$$\begin{aligned}
M_{T_0} =& b_{T_0}(T_0^{\frac{1}{q}} + a_{T_0}T_0)(||S(\ )u_0||_{T_0,p} + \varepsilon) \\
&+ \{b_{T_0}c_{T_0} + r_2(0,0)T_0) + ||G(0,0,0)||\}T_0
\end{aligned}$$

from (3.1). We define

$$K = \{u \in L^p(0,T_0;X) \mid ||u - S(\cdot)u_0||_{T_0,p} \le \varepsilon\}.$$

Then K is a non-empty closed convex subset of $L^p(0,T_0;X)$. From the method employed Vrabie([9]) and Shioji([7]), we define an operator $Q : K \to L^p(0,T_0;X)$ by $Qu = v$ for every $u \in K$, where $v(t)$ is the unique integral solution of

$$(3.3) \qquad \begin{cases} \dfrac{dv(t)}{dt} + Av(t) \ni G(t,u(t),L_tu), \quad 0 \le t \le T_0, \\ v(0) = u_0, \end{cases}$$

from (3.2) and Proposition 1.

We will show $Q(K) \subset K$. Let $u \in K$. Since $S(t)u_0$ is the unique integral solution of

$$\begin{cases} \dfrac{dv(t)}{dt} + Av(t) \ni 0, \quad 0 \le t \le T_0, \\ v(0) = u_0, \end{cases}$$

from Proposition 1 with (3.3), for every $t \in [0,T_0]$,

$$||(Qu)(t) - S(t)u_0|| \le \int_0^{T_0} ||G(\tau,u(\tau),L_\tau u)||d\tau$$

and thus we have from (3.2),

$$\begin{aligned} ||Qu - S(\cdot)u_0||_{T_0,p} &\le (\int_0^{T_0} (\int_0^{T_0} ||G(\tau,u(\tau),L_\tau u)||d\tau)^p dt)^{\frac{1}{p}} \\ &\le T_0^{\frac{1}{p}} \int_0^{T_0} ||G(\tau,u(\tau),L_\tau u)||d\tau \le T_0^{\frac{1}{p}} M_{T_0} \le \varepsilon. \end{aligned}$$

Since $Qu \in C([0,T_0];X)$, $Qu \in K$. Thus $Q(K) \subset K$.

Next, we will show that $Q : K \to K$ is continuous in $L^p(0,T_0;X)$. Let $w, z \in K$. Since, from Proposition 1, for every $t \in [0,T_0]$,

$$||(Qw)(t) - (Qz)(t)|| \le \int_0^{T_0} ||G(\tau,w(\tau),L_\tau w) - G(\tau,z(\tau),L_\tau z)||d\tau,$$

we have, from (L1) and (G1),

$$
\begin{aligned}
&\|Qw - Qz\|_{T_0,p} \\
&\le T_0^{\frac{1}{p}} \int_0^{T_0} \|G(\tau, w(\tau), L_\tau w) - G(\tau, z(\tau), L_\tau z)\| d\tau \\
&\le b_{T_0} T_0^{\frac{1}{p}} \int_0^{T_0} (\|w(\tau) - z(\tau)\| + a_{T_0}\|w - z\|_{\tau,p}) d\tau \\
&\le b_{T_0} T_0^{\frac{1}{p}} (T_0^{\frac{1}{q}} + a_{T_0} T_0) \|w - z\|_{T_0,p}.
\end{aligned}
\tag{3.4}
$$

Thus $Q : K \to K$ is continuous in $L^p(0, T_0; X)$. On the other hand, from compactness of $\{S(t) \mid t \ge 0\}$, (3.2) and Proposition 3, $\{Qu \mid u \in K\}$ is relatively compact in $L^p(0, T_0; X)$. Hence, from Schauder's fixed point theorem, there exists $u \in K$ such that $Qu = u$. Thus $u(t)$ is a local integral solution of (1.1).

THEOREM 2. *Assume that* (A), (L) *and* (G) *are satisfied and the nonlinear semigroup* $\{S(t) \mid t \ge 0\}$ *on* $\overline{D(A)}$ *generated by* $-A$ *is compact. Let* U *be an open subset of* X. *Let* $u_0 \in \overline{D(A)} \cap U$. *Then* (1.1) *has a local integral solution in* U.

Proof. Let $\delta > 0$. Let $u \in L^\infty(0, T; U)$ with $\|u\|_{T,\infty} = esssup_{0 \le t \le T} \|u(t)\| \le \delta$. Since $\|u\|_{T,p} = (\int_0^T \|u(t)\|^p dt)^{\frac{1}{p}} \le \delta T^{\frac{1}{p}}$ for every $u \in L^p(0, T; U)$, from (3.1), we have for $0 < h < T$,

$$
\begin{aligned}
&\int_0^h \|G(t, u(t), L_t u)\| dt \\
&\qquad \le b_h(h^{\frac{1}{q}} + a_h h)\delta h^{\frac{1}{p}} + \{b_h c_h + r_2(0,0)h + \|G(0,0,0)\|\} h
\end{aligned}
$$

and thus for every $\delta > 0$,

$$
\lim_{h \to 0+} \int_0^h \|G(t, u(t), L_t u)\| dt = 0 \tag{3.5}
$$

uniformly for $u \in L^\infty(0, T; U)$ with $\|u\|_{T,\infty} \le \delta$.

Since U is open and $\lim_{t \to 0+} S(t)u_0 = u_0$ with (3.5), there exist $\varepsilon > 0$ and $0 < T_0 \le T$ such that $x \in U$ for every $x \in X$ with

$$
\|x - u_0\| \le \varepsilon + \max_{0 \le t \le T_0} \|S(t)u_0 - u_0\|,
$$

and

$$\int_0^{T_0} ||G(t, u(t), L_t u)|| dt \le \varepsilon \tag{3.6}$$

uniformly for $u \in L^\infty(0, T_0; U)$ with $||u||_{T_0,\infty} \le \delta$ where $\delta = \epsilon + \max_{0\le t\le T_0} ||S(t)u_0||$. We define

$$K = \{u \in L^\infty(0, T_0; U) \mid ||u - S(\cdot)u_0||_{T_0,\infty} \le \varepsilon\}.$$

Then K is a non-empty closed convex subset of $L^\infty(0, T_0; U)$ with respect to $||\cdot||_{T_0,\infty}$. We define $Q : K \to L^p(0, T_0; X)$ by the same method in the proof of Theorem 1. We will prove $Q(K) \subset K$ and $Q : K \to K$ is continuous in $L^p(0, T_0; U)$. Let $u \in K$. Then $||u||_{T_0,\infty} \le \delta$. From Proposition 1 and (3.6), for every $t \in [0, T_0]$,

$$||(Qu)(t) - S(t)u_0|| \le \int_0^{T_0} ||G(t, u(t), L_t u)|| dt \le \varepsilon,$$

from which for every $t \in [0, T_0]$,

$$\begin{aligned} ||(Qu)(t) - u_0|| &\le ||(Qu)(t) - S(t)u_0|| + ||S(t)u_0 - u_0|| \\ &\le \varepsilon + \max_{0\le t\le T_0} ||S(t)u_0 - u_0||. \end{aligned}$$

Thus $Qu \in K$ and $Q(K) \subset K$. From (3.4), Q is continuous with respect to $||\cdot||_{T_0,\infty}$. From (3.6), Proposition 3 and Schauder's fixed point theorem, there exists a fixed point $u \in K$ of Q. This u is a local integral solution of (1.1).

Next, we consider the local existence theorem under the assumption of compactness of the resolvent.

THEOREM 3. *Assume that* (A), (L) *and* (G) *are satisfied and the resolvent* $J_\lambda = (I + \lambda A)^{-1}$ *of* A *for every* $\lambda > 0$ *is compact. Let* U *be an open subset of* X *and* $\{S(t) \mid t \ge 0\}$ *a nonlinear semigroup on* $\overline{D(A)}$ *generated by* $-A$. *Put*

$$B = \{u \in L^\infty(0, T_0; U) \mid ||u||_{T_0,\infty} \le \delta\}$$

where $\varepsilon > 0$, $0 < T_0 \le T$, $\delta > 0$ *are as in the proof of* Theorem 2. *Assume that*

$$\lim_{h\to 0+}\int_0^{T_0-h}\|u(t+h)-u(t)\|dt = 0 \tag{3.7}$$

uniformly for $u \in B$. *Let* $u_0 \in \overline{D(A)} \cap U$. *Then* (1.1) *has a local integral solution.*

Proof. Define K as in the proof of Theorem 2. Let $u \in B$. Then for every $h > 0$ with $t + h \le T_0$, we have, from (L2) and (G),

$$\begin{aligned}
&\|G(t+h,u(t+h),L_{t+h}u) - G(t,u(t),L_tu)\| \\
&\quad \le \|G(t+h,u(t+h),L_{t+h}u) - G(t+h,u(t),L_tu)\| \\
&\qquad + \|G(t+h,u(t),L_tu) - G(t,u(t),L_tu)\| \\
&\quad \le b_{T_0}(\|u(t+h)-u(t)\| + \|L_{t+h}u - L_tu\|) + hr_2(\|u(t)\|,\|L_tu\|) \\
&\quad \le b_{T_0}(\|u(t+h)-u(t)\| + hr_1(\|u\|_{T_0,1})) + hr_2(\delta, a_{T_0}\|u\|_{T_0,p} + c_{T_0}) \\
&\quad \le b_{T_0}(\|u(t+h)-u(t)\| + hr_1(\delta T_0) + hr_2(\delta, a_{T_0}\delta T_0 + c_{T_0})
\end{aligned}$$

and thus

$$\begin{aligned}
&\int_0^{T_0-h}\|G(t+h,u(t+h),L_{t+h}u) - G(t,u(t),L_tu)\|dt \\
&\qquad \le b_{T_0}\int_0^{T_0-h}\|u(t+h)-u(t)\|dt + \alpha_{T_0,h},
\end{aligned} \tag{3.8}$$

where

$$\alpha_{T_0,h} = T_0\{b_{T_0}hr_1(\delta T_0) + hr_2(\delta, a_{T_0}\delta T_0 + c_{T_0})\}$$

and $\lim_{h\to 0+}\alpha_{T_0,h} = 0$. From (3.7) and (3.8),

$$\lim_{h\to 0+}\int_0^{T_0-h}\|G(t+h,u(t+h),L_{t+h}u) - G(t,u(t),L_tu)\|dt = 0 \tag{3.9}$$

uniformly for $u \in B$. Let $\{S(t) \mid t \ge 0\}$ be the nonlinear semigroup on $\overline{D(A)}$ generated by $-A$. Choose $0 < T_1 \le T_0$ satisfying $T_1 b_{T_1} < 1$. We define

$$K_1 = \{u \in K \mid \int_0^{T_1-h}\|u(t+h)-u(t)\|dt \le \gamma(h) \text{ for every } 0 < h < T_1\},$$

where

$$\gamma(h) = \frac{T_1}{1 - T_1 b_{T_1}} \{ \sup_{u \in K_1} \int_0^h ||G(t, u(t), L_t u)||dt + ||S(h)u_0 - u_0|| + \alpha_{T_1,h} \}.$$

Then K_1 is a non-empty closed convex subset of $L^\infty(0, T_1; U)$ with respect to $||\cdot||_{T_1,1}$. We define $Q : K_1 \to L^1(0, T_1; X)$ by the same way in the proof of Theorem 1. We will show $Q(K_1) \subset K_1$. Let $u \in K_1$. From Proposition 2 and (3.8), for $0 < h < T_1$,

$$\begin{aligned}
&||(Qu)(t+h) - (Qu)(t)|| \\
&\quad \le \int_0^h ||G(\tau, u(\tau), L_\tau u)||d\tau + ||S(h)u_0 - u_0|| \\
&\qquad + \int_0^{T_1 - h} ||G(\tau + h, u(\tau + h), L_{\tau+h} u) - G(\tau, u(\tau), L_\tau u)||d\tau \\
&\quad \le \sup_{u \in K_1} \int_0^h ||G(\tau, u(\tau), L_\tau u)||d\tau + ||S(h)u_0 - u_0|| \\
&\qquad + b_{T_1} \int_0^{T_1 - h} ||u(\tau + h) - u(\tau)||d\tau + \alpha_{T_1,h},
\end{aligned}$$

and thus from (3.10)

$$\begin{aligned}
&\int_0^{T_1 - h} ||(Qu)(t+h) - (Qu)(t)||dt \\
&\qquad \le T_1 \{ \sup_{u \in K_1} \int_0^h ||G(t, u(t), L_t u)||dt + ||S(h)u_0 - u_0|| \\
&\qquad\quad + b_{T_1} \int_0^{T_1 - h} ||u(t+h) - u(t)||dt + \alpha_{T_1,h} \} \\
&\qquad \le T_1 \{ \sup_{u \in K_1} \int_0^h ||G(t, u(t), L_t u)||dt + ||S(h)u_0 - u_0|| \\
&\qquad\quad + b_{T_1} \gamma(h) + \alpha_{T_1,h} \} \\
&\qquad = \gamma(h).
\end{aligned}$$

Hence $Qu \in K_1$ and thus $Q(K_1) \subset K_1$. From (3.4), $Q : K_1 \to K_1$ is continuous. On the other hand, from compactness of $J_\lambda = (I + \lambda A)^{-1}$

for every $\lambda > 0$, (3.5), (3.9) and Proposition 4, $\{Qu \mid u \in K_1\}$ is relatively compact in $L^1(0, T_1; U)$. Therefore, from Schauder's fixed point theorem, there exists a fixed point $u \in K_1$. This u is a local integral solution of (1.1) in U.

4. Example

Let Ω be a subset of $\mathcal{R}^n$ with its boundary Γ. Let $T > 0$. We consider the nonlinear integro-differential equation

$$\tag{4.1} \begin{cases} \dfrac{\partial u(t,x)}{\partial t} - \Delta\beta(u(t,x)) = \displaystyle\int_0^t k(t,s,u(s,x))ds, \\ \qquad 0 \le t \le T, \quad x \in \Omega, \\ \beta(u(t,x)) = 0, \quad 0 \le t \le T,\ x \in \Gamma, \\ u(0,x) = u_0(x), \quad x \in \Omega, \end{cases}$$

where $\beta : \mathcal{R} \to \mathcal{R}$, $k : [0,T] \times [0,T] \times \mathcal{R} \to \mathcal{R}$ and $u_0 : \Omega \to \mathcal{R}$.

THEOREM 4. *Let Ω be a bounded open subset of $\mathcal{R}^n$ for $n \ge 2$ with sufficiently smooth boundary Γ. Let $\beta \in C(\mathcal{R}) \cap C^1(\mathcal{R} - \{0\})$ with $\beta(0) = 0$ and for every $c \in \mathcal{R} - \{0\}$, $\beta'(c) \ge b|c|^{a-1}$ for some $a > (n-2)/n$ and $b > 0$. Let $k : [0,T] \times [0,T] \times \mathcal{R} \to \mathcal{R}$ satisfying*

$$\tag{4.2} |k(t,s,u)| \le M$$

for every $t, s \in [0,T]$, $u \in \mathcal{R}$ and for some $M > 0$, and

$$\tag{4.3} |k(t,s,u) - k(\tau,\sigma,v)| \le \alpha(|t-\tau| + |s-\sigma| + |u-v|)$$

for every $t, s, \tau, \sigma \in [0,T]$, $u, v \in \mathcal{R}$ and for some $\alpha > 0$ Let $u_0 \in L^1(\Omega)$. Then (4.1) has a local integral solution.

Proof. We put $X = L^1(\Omega)$ with norm $\|\cdot\|$. We define an opertor $A : D(A) \subset X \to X$ by

$$D(A) = \{u \in X \mid \beta(u) \in W_0^{1,1}(\Omega), \quad \Delta\beta(u) \in X\},$$

$$Au = -\Delta\beta(u) \quad \text{for every} \quad u \in D(A).$$

Then, from Lemma 2.6.2 of Vrabie([8]), A is m-accretive and $-A$ generates a compact semigroup on $\overline{D(A)} = X$. We put for every $t \in [0,T]$ and $x \in \Omega$,

$$(L_t u)(x) = \int_0^t k(t,s,u(s,x))ds \tag{4.4}$$

for every $u \in L^1([0,t] \times \Omega) \subset L^1(0,t;X)$. Put $z(t)(x) = u(t,x)$. Then (4.1) can be formulated as

$$\begin{cases} \dfrac{dz(t)}{dt} + Az(t) = L_t z, \quad 0 \le t \le T, \\ z(t) = u_0, \end{cases} \tag{4.5}$$

in X. Then, from (4.3) and (4.4), for every $t \in [0,T]$ and $u, v \in L^1([0,t] \times \Omega)$,

$$\begin{aligned} ||L_t u - L_t v|| &\le \alpha \int_\Omega \int_0^t |u(s,x) - v(s,x)| ds dx \\ &= \alpha \int_0^t \int_\Omega |u(s,x) - v(s,x)| dx ds \\ &= \alpha ||u - v||_{t,1}, \end{aligned}$$

and from (4.2)–(4.4), for every $t, s \in [0,T]$ and $u \in L^1([0,t] \times \Omega)$,

$$\begin{aligned} ||L_t u - L_s u|| &= \int_\Omega \int_0^t |k(t,\tau,u(\tau,x)) - k(s,\tau,u(\tau,x))| d\tau dx \\ &\quad + \int_\Omega \int_s^t |k(s,\tau,u(\tau,x))| d\tau dx \\ &\le (\alpha T + M) m(\Omega) |t - s|. \end{aligned}$$

Hence, from Theorem 1, (4.5) has a local integral solution $z(t)$. Thus (4.1) has a local integral solution $u(t,x)$.

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