

ON LANDSBERG SPACES OF DIMENSION TWO WITH A DECOMPOSITION OF CUBIC METRIC

IL-YONG LEE

1. Introduction

The (α, β) -metric is a Finsler metric which is constructed from a Riemannian metric α and a differential 1-form β

If the covariant derivative $C_{h_{ij}|k}$ of the C -torsion tensor in the Cartan connection satisfies $C_{h_{ij}|k}y^k = 0$, then we define a Finsler space as a *Landsberg space*. In the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x, y)$ satisfies $I_{|i}y^i = 0$.

A Berwald space is characterized by $C_{h_{ij}|k} = 0$. Berwald spaces are specially interesting and important, because the connection is linear, and many examples of Berwald spaces have been known to exist.

The purpose of this paper is to find two-dimensional Landsberg spaces with a decomposition of cubic metric. First we are trying to investigate the condition for the Finsler space with its metric to be a Berwald space. Next we determine the difference vector and the main scalar with the above metric. Finally we derive the conditions that a two-dimensional Finsler space with the above metric be a Landsberg space and show that a two-dimensional Finsler space with its metric which is a Landsberg space is a Berwald space. Here the difference vector and the main scalar play the principal role in the present paper.

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2. Preliminaries

Let $F^n = (M^n, L(x, y))$ be an n -dimensional Finsler space with a fundamental metric function $L(x, y)$. The metric is called an (α, β) -metric if L is a (1) p -homogeneous function $L(\alpha, \beta)$ [7] of α and β , where $\alpha = \sqrt{a_{ij}(x)y^i y^j}$ is a Riemannian metric in the underlying manifold M^n and $\beta = b_i(x)y^i$ is a differential 1-form in M^n . In the following the Riemannian metric α is *not supposed to be positive-definite* and we shall restrict our discussions to a domain of (x, y) , where β does not vanish. The covariant differentiation in the Levi-Civita connection $\{\gamma_j^i k(x)\}$ of R^n is denoted by the semi-colon. Let us list the symbols here for the late use as follows:

$$\begin{aligned} b^i &= a^{ir} b_r, & b^2 &= a^{rs} b_r b_s, \\ 2r_{ij} &= b_{i,j} + b_{j,i}, & 2s_{ij} &= b_{i,j} - b_{j,i}, \\ r_j^i &= a^{ir} r_{rj}, & s_j^i &= a^{ir} s_{rj}, & r_i &= b_r r_i^r, & s_i &= b_r s_i^r \end{aligned}$$

The following Lemma has been shown as follows:

LEMMA 2.1 [2, 5]. *If $\alpha^2 \equiv 0 \pmod{\beta}$, that is, $a_{ij}(x)y^i y^j$ contains $b_i(x)y^i$ as factor, then the dimension n is equal to two and b^2 vanishes. In this case we have $\delta = d_i(x)y^i$ satisfying $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

LEMMA 2.2 [5]. *We consider the two-dimensional case.*

(1) *If $b^2 \neq 0$, then there exist a sign $\varepsilon = \pm 1$ and $\delta = d_i(x)y^i$ and such that $\alpha^2 = \beta^2/b^2 + \varepsilon\delta^2$ and $d_i b^i = 0$.*

(2) *If $b^2 = 0$, then there exists $\delta = d_i(x)y^i$ such that $\alpha^2 = \beta\delta$ and $d_i b^i = 2$.*

If there are two functions $f(x), g(x)$ satisfying $f\alpha^2 + g\beta^2 = 0$, then $f = g = 0$ is obvious, because $f \neq 0$ implies a contradiction $\alpha^2 = (-g/f)\beta^2$.

LEMMA 2.3 [5]. *We consider the two-dimensional case.*

(1) *If there exist two functions $f(x), g(x)$ satisfying $f\beta^2 + g\gamma^2 = 0$, then $f = g = 0$, or $f = g, b^2 = 0$.*

(2) *If there exist two 1-forms λ, μ satisfying $\lambda\alpha^2 + \mu\beta^2 = 0$, then*

$$(1^\circ) \quad b^2 \neq 0 : \lambda = \mu = 0, \quad (2^\circ) \quad b^2 = 0 : \lambda = f\beta, \mu = -f\delta,$$

where $f = \dot{f}(x)$ and δ is the one in (2) of Lemma 2.2.

3. The condition to be the Berwald space

From now on, in the present paper, we deal with the special (α, β) -metric L satisfying $L^3(\alpha, \beta) = \alpha^2\beta$. This metric is called a *decomposition of cubic metric*.

Let $F^n = (M^n, L(\alpha, \beta))$ be an n -dimensional Finsler space with an (α, β) -metric given by

$$(3.1) \quad L^3(\alpha, \beta) = \alpha^2\beta.$$

Then we obtain

$$(3.2) \quad 3L^2L_\alpha = 2\alpha\beta, \quad 3L^2L_\beta = \alpha^2, \quad 9L^2L_{\alpha\alpha} = -2\beta.$$

Since $B\Gamma$ is L -metrical, $\partial_i L - G_i^k \hat{\partial}_k L = 0$ is rewritten as follows:

$$(3.3) \quad L_\alpha B_j^k y^j y_k = \alpha L_\beta (b_{j,i} - B_j^k b_k) y^j,$$

where $y_k = a_{ki} y^i$. In the following the raising and lowering of indices are done by means of the Riemannian $a_{ij}(x)$. Substituting (3.2) in (3.3), we obtain

$$(3.4) \quad \alpha^2 (b_{j,i} - B_j^k b_k) y^j - 2\beta B_j^k y^j y_k = 0.$$

Now, we assume that the Finsler space F^n with an (α, β) -metric given by (3.1) is a Berwald space, that is, $G_j^i k$ is a function of the position alone. Then the left-hand side of the above equation is a polynomial of three order in (y^j) and shows the existence of function $p_i(x)$ satisfying

$$B_j^k y^j y_k = p_i \alpha^2, \quad (b_{j,i} - B_j^k b_k) y^j = 2p_i \beta.$$

The former is written as $B_j^k a_{hk} + B_h^k a_{kj} = 2p_i a_{jh}$, which implies

$$(3.5) \quad B_j^k = p_i \delta_j^k + p_j \delta_i^k - p^k a_{ij}.$$

Therefore the latter gives

$$(3.6) \quad b_{j,i} = 3p_i b_j + p_j b_i - p_b a_{ij},$$

where $p_b = p_i b^i$.

Conversely, if there exists the vector $p_i(x)$ satisfying (3.6), we have $L|_i = 0$ with respect to $G_j^i k = \gamma_j^i k + B_j^i k$, where $B_j^i k$ is given by (3.5). Hence, by the well-known Hashiguchi-Ichijyō's Theorem [2], the Finsler space is a Berwald space.

Thus we have the following theorem.

THEOREM 3.1. *Let F^n be the Finsler space with an (α, β) -metric given by (3.1). Then F^n is a Berwald space if and only if there exists the covariant vector $p_i(x)$ satisfying (3.6).*

4. The difference vector

The Berwald connection $B\Gamma = \{G_j^i k, G_j^i, 0\}$ of F^n plays one of the leading roles in the present paper. Denote by $B_j^i k$ the difference tensor [8] of $G_j^i k$ from $\gamma_j^i k$ as follows:

$$G_j^i k(x, y) = \gamma_j^i k(x) + B_j^i k(x, y).$$

With the subscript 0, transvection by y^i , we obtain

$$G_j^i = \gamma_0^i j + B_j^i, \quad 2G^i = \gamma_0^i 0 + 2B^i,$$

where $B_j^i = \hat{\partial}_j B^i$ and $B_j^i k = \hat{\partial}_k B_j^i$.

It is noted that the Cartan connection also has the nonlinear connection $\{G_j^i\}$ common to $B\Gamma$. $B^i(x, y)$ is called the *difference vector* in the present paper.

By M. Matsumoto [8] the difference vector $B^i(x, y)$ in $2G^i = \gamma_0^i 0 + 2B^i$ is given by

$$(4.1) \quad B^i = Ee^i + (\alpha L_\beta / L_\alpha) s_0^i - (\alpha L_{\alpha\alpha} / L_\alpha)(C + \alpha r_{00} / 2\beta) c^i,$$

where quantities C and E are determined by

$$(4.2) \quad C = -(\alpha^2 L_\beta / \beta L_\alpha) s_0 - (\alpha^2 b^2 - \beta^2)(\alpha L_{\alpha\alpha} / \beta^2 L_\alpha)(C + \alpha r_{00} / 2\beta),$$

$$(4.3) \quad L_\beta r_{00} = 2LE / \alpha - 2\beta L_\beta C / \alpha,$$

where $c_i = e_i - (\alpha/\beta)b_i$, $e_i = y_i/\alpha$, $E = B^i e_i$ and $C = B^i c_i$. Substituting (3.2) in (4.2), we have

$$(4.4) \quad C = \frac{\alpha\gamma^2}{2\beta\Omega} r_{00} - \frac{3\alpha^3}{2\Omega} s_0,$$

where we put $\Omega = 3\beta^2 - \gamma^2$.

Substituting (3.1) and (3.2) in (4.3), we get

$$(4.5) \quad E = \frac{\alpha\beta}{2\Omega}r_{00} - \frac{\alpha^3}{2\Omega}s_0.$$

Substituting (3.2), (4.4) and (4.5) in (4.1), we have

$$(4.6) \quad B^i = \frac{1}{2\Omega}(\beta r_{00} - \alpha^2 s_0) \left(2y^i - \frac{\alpha^2}{\beta} b^i \right) + \frac{\alpha^2}{2\beta} s_0^i.$$

Thus we obtain the following theorem.

THEOREM 4.1. *Let F^n be the Finsler space with an (α, β) -metric given by (3.1). Then the difference vector $B^i(x, y)$ in $2G^i = \gamma_0^i s_0 + 2B^i$ is given by (4.6).*

5. The main scalar

We are to treat the main scalar I of a two-dimensional Finsler space with an (α, β) -metric. By the paper [6] the main scalar I of a two-dimensional Finsler space with an (α, β) -metric is given by

$$(5.1) \quad \epsilon I^2 = \frac{L^4 \gamma^2 (T_2)^2}{4\alpha^4 (T)^3},$$

where

$$(5.2) \quad T = \left(\frac{L}{\alpha} \right)^3 (L_\alpha + w\alpha\gamma^2), \quad T_2 = \frac{\partial T}{\partial \beta},$$

$$w = \frac{L_{\alpha\alpha}}{\beta^2} = -\frac{L_{\alpha\beta}}{\alpha\beta} = \frac{L_{\beta\beta}}{\alpha^2}.$$

Substituting (3.1) and (3.2) in (5.2) and using $9\beta L^2 w = -2$, we have

$$(5.3) \quad T = \frac{2\Omega}{9L^2}$$

Differentiating (5.3) with respect to β , we get

$$(5.4) \quad T_2 = \frac{4(9\beta^2 + \gamma^2)}{27\beta L^2}.$$

Substituting (3.1), (5.3) and (5.4) in (5.1), we obtain

$$(5.5) \quad \epsilon I^2 = \frac{\gamma^2(9\beta^2 + \gamma^2)^2}{2\Omega^3}.$$

Therefore we get the following theorem.

THEOREM 5.1. *The main scalar I of a two-dimensional Finsler space with (3.1) is given by (5.5).*

6. Finsler spaces with (3.1) which are Landsberg spaces

Before dealing with Finsler spaces with (3.1) which are Landsberg spaces, we investigate to play the leading roles in this section.

Since $B\Gamma$ is L -metrical, $L(\alpha, \beta)$ satisfies

$$L_{|i} = \partial_i L - (\partial_r L) G_i^r = 0 = L_1 \alpha_{|i} + L_2 \beta_{|i},$$

where $(L_1, L_2) = (\partial L / \partial \alpha, \partial L / \partial \beta)$, and so

$$(6.1) \quad \alpha_{|i} = -\frac{L_2}{L_1} \beta_{|i}.$$

It is observed that $\beta_{|i} = b_{s|i} y^s = (b_{s,i} - b_r B_s^r) y^s$, which implies

$$(6.2) \quad \beta_{|i} y^i = r_{00} - 2b_r B^r.$$

For the scalar b^2 we have $b_{|i}^2 y^i = (\partial_i b^2) y^i = b_{,i}^2 y^i = 2b^r (r_{r1} + s_{r1}) y^i$, which shows

$$(6.3) \quad b_{|i}^2 y^i = 2(r_0 + s_0).$$

Next the quadratic form

$$\gamma^2 = b^2 \alpha^2 - \beta^2 = (b^2 a_{ij} - b_i b_j) y^i y^j,$$

plays a role in the following. From the equations above it is easy to show

$$(6.4) \quad \gamma_{|i}^2 y^i = 2(r_0 + s_0) \alpha^2 - 2 \left(\frac{L_2}{L_1} b^2 \alpha + \beta \right) (r_{00} - 2b_r B^r).$$

We consider a Finsler space $F^n = (M^n, L^3 = \alpha^2 \beta)$. By (4.6) the difference vector of F^n is given by

$$(6.5) \quad 2B^i = \frac{1}{\Omega} (\beta r_{00} - \alpha^2 s_0) \left(2y^i - \frac{\alpha^2}{\beta} b^i \right) + \frac{\alpha^2}{\beta} s_0^i.$$

The main scalar of a two-dimensional Finsler space with (3.1) is given by (5.5) as follows:

$$(6.6) \quad \varepsilon I^2 = \frac{\gamma^2(9\beta^2 + \gamma^2)^2}{2\Omega^3}.$$

Before discussing our problem, we have to consider the assumption $\Omega \neq 0$ in the two-dimensional case, because Ω appears in the denominators in (6.5) and (6.6). Lemma 2.3 shows that $\Omega = 3\beta^2 - \gamma^2$ does not vanish because of $3 \neq -1$. Consequently $\Omega \neq 0$ is a proper assumption in the present section.

Now we deal with the condition that a two-dimensional Finsler space with (3.1) be a Landsberg space. It follows from (6.5) that

$$r_{00} - 2b_r B^r = \frac{2\beta}{\Omega}(\beta r_{00} - 2\alpha^2 s_0).$$

In the two-dimensional case, a general Finsler space is a Landsberg space, if and only if its main scalar $I(x, y)$ satisfies $I_{|i} y^i = 0$ (see [7]). Canceling the denominator of (6.6), the covariant differentiation leads to

$$\varepsilon I_{|i}^2 = \frac{27}{2\Omega^4}(\beta^2 + \gamma^2)(9\beta^2 + \gamma^2)(\beta^2 \gamma_{|i}^2 - \gamma^2 \beta_{|i}^2).$$

Thus, transvection by y^i , we have

$$\varepsilon I_{|i}^2 y^i = \frac{27}{2\Omega^4}(\beta^2 + \gamma^2)(9\beta^2 + \gamma^2)(\beta^2 \gamma_{|i}^2 y^i - \gamma^2 \beta_{|i}^2 y^i).$$

Consequently the two-dimensional Finsler space with (3.1) is a Landsberg space, if and only if

$$(\beta^2 + \gamma^2)(9\beta^2 + \gamma^2)(\beta^2 \gamma_{|i}^2 y^i - \gamma^2 \beta_{|i}^2 y^i) = 0.$$

Hence the following three cases should be considered to find the condition.

(Case A) $\beta^2 + \gamma^2 = b^2 \alpha^2 = 0$.

This shows $b^2 = 0$. (6.6) gives $\varepsilon I^2 = -\frac{1}{2}$ and Lemma 2.2 shows $L^3 = \beta^2 \delta$. Since I becomes constant, the space is a Berwald space from the well-known Berwald's theorem ([1], [7]). Therefore we have the following theorem.

THEOREM 6.1. *Let F^2 be a Finsler space of dimension two with $L^3 = \alpha^2\beta$. If b^2 of F^2 vanishes, then F^2 is a Berwald space. Then L is written as $L^3 = \beta^2\delta$, where δ is the one in (2) of Lemma 2.2 and the main scalar I is such that $\varepsilon I^2 = -\frac{1}{2}$.*

(Case B) $9\beta^2 + \gamma^2 = 0$:

Lemma 2.3 shows immediately a contradiction because of $9 \neq 1$. Thus $9\beta^2 + \gamma^2 \neq 0$.

(Case C) $\beta^2\gamma_{|i}^2y^i - \gamma^2\beta_{|i}^2y^i = 0$:

By means of (6.2), (6.4) and (6.5) this equation is written as follows:

$$(6.7) \quad 3(2\beta s_0 + \beta r_0 - b^2 r_{00})\beta - (r_0 - 2s_0)\gamma^2 = 0.$$

First this gives a condition $(r_0 - 2s_0)\gamma^2 \equiv 0 \pmod{\beta}$. Since $b^2 \neq 0$ may be supposed in this case, Lemma 2.1 shows $\gamma^2 \not\equiv 0 \pmod{\beta}$ and so there exists a function $g(x)$ satisfying

$$(6.8) \quad r_0 - 2s_0 = g\beta.$$

Then (6.7) is reduced to

$$(6s_0 + 3r_0 + g\beta)\beta - (3r_{00} + g\alpha^2)b^2 = 0.$$

This implies that there exists a 1-form $\mu = m_i(x)y^i$ such that

$$(6.9) \quad 3r_{00} + g\alpha^2 = \beta\mu,$$

and the above is reduced to

$$6s_0 + 3r_0 + g\beta = b^2\mu.$$

The above equation and (6.8) yield

$$(6.10) \quad s_0 = \frac{1}{12}(b^2\mu - 4g\beta),$$

$$(6.11) \quad r_0 = \frac{1}{6}(b^2\mu + 2g\beta).$$

Consequently (6.9), (6.10) and (6.11) are obtained from (6.7). Since (6.9) is written in the form

$$r_{ij} + \frac{1}{3}ga_{ij} = \frac{1}{6}(m_i b_j + m_j b_i),$$

the transvection by $b^i y^j$ yields

$$r_0 + \frac{1}{3}g\beta = \frac{1}{6}(b^2\mu + m_b\beta),$$

where $m_b = m_i b^i$.

Comparing the above with (6.11), we have $m_b = 4g$. Thus we get the condition in the form

$$r_{00} = -\frac{1}{12}m_b\alpha^2 + \frac{1}{3}\beta\mu, \quad s_0 = -\frac{1}{12}(m_b\beta - b^2\mu).$$

Eliminating μ from the above, the condition for the space to be a Landsberg space is written as follows:

$$(6.12) \quad r_{00} = -\frac{1}{12}f\alpha^2 + \frac{1}{3b^2}f\beta^2 + \frac{4}{b^2}\beta s_0,$$

where $f(x) = m_b$. Thus we have the following theorem.

THEOREM 6.2. *The condition that a twodimensional Finsler space with $L^3(\alpha, \beta) = \alpha^2\beta$ be a Landsberg space is given by (6.12).*

Now we shall show the following theorem.

THEOREM 6.3. *Let F^2 be a two-dimensional Finsler space with $L^3 = \alpha^2\beta$. If F^2 with $b^2 \neq 0$ is a Landsberg space, then F^2 is a Berwald space.*

Proof. The condition (3.6) to be a Berwald space may be written in the form

$$(6.13) \quad \begin{aligned} (1) \quad r_{ij} &= 2(p_i b_j + p_j b_i) - p_b a_{ij}, \\ (2) \quad s_{ij} &= p_j b_i - p_i b_j. \end{aligned}$$

Now, let F^2 be a Landsberg space, that is, suppose that (6.12) holds. Thus the system of linear equations

$$b^1 p_1 + b^2 p_2 = \frac{f}{12}, \quad -b_2 p_1 + b_1 p_2 = s_{12}.$$

The latter is nothing but (2) of (6.13). Then we get

$$s_0 = b^2 \phi - p_b \beta, \quad \phi = p_1(x)y^2,$$

and (6.12) is now written as

$$r_{00} = 4\beta\phi - p_b \alpha^2,$$

which is nothing but (1) of (6.13). Thus the proof is completed.

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Department of Mathematics
 Kyungsung University
 Pusan 608-736, Korea
 E-mail : iylee@star.kyungsung.ac.kr