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L^p ESTIMATES FOR AREA INTEGRALS WITH RESPECT TO SINGULAR MEASURES

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1. Introduction

The theory of the tent spaces on the upper half-space $\mathbb{R}^{n+1}_+$ was introduced from the work of R. R. Coifman, Y. Meyer and E. M. Stein [1]. Their works resulted in many applications involving the study of a variety of questions related to harmonic analysis. We carry out the theory of the tent spaces on the generalized upper half-space $X\times(0,\infty)$, where X is a space of homogeneous type.

We begin by introducing the notion of a space of homogeneous type [2]: Let X be a topological space endowed with Borel measure μ Assume that d is a pseudo-metric on X, that is, a nonnegative function on $X \times X$ satisfying

(i) $d(x,x)=0; d(x,y)>0$ if $x\neq y$,
(ii) $d(x,y)=d(y,x)$, and
(iii) $d(x,z)\leq K(d(x,y)+d(y,z))$, where K is some fixed constant.

Assume further that

(a) the balls $B(x,\rho)=\{y\in X \cdot d(x,y)<\rho\}$, $\rho>0$, form a basis of open neighborhoods at $x\in X$,

and that μ satisfies the doubling property:

(b) $0<\mu(B(x,2\rho))\leq A\mu(B(x,\rho))<\infty$, where A is some fixed constant.

Then we call (X,d,μ) a *space of homogeneous type.*

Property (iii) will be referred to as the "triangle inequality." Note that property (b) implies that for every $C>0$ there exists a constant $A_C<\infty$ such that

$$\mu(B(x,C\rho))\leq A_C\mu(B(x,\rho))$$

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for all $x \in X$ and $\rho > 0$.

Now consider the space $X \times (0, \infty)$, which is a kind of generalized upper half-space over X. We then define the analogue of nontangential or conoical regions as follows. For $x \in X$, set

$$\Gamma(x) = \{(y,t) \in X \times (0,\infty) : x \in B(y,t)\}.$$

For any set $E \subset X$, the *tent* over E is the set

$$\hat{E} = \{(y,t) \in X \times (0,\infty) : B(y,t) \subset E\}.$$

It is then very easy to check that

$$\hat{E} = (X \times (0,\infty)) \setminus \bigcup_{x \notin E} \Gamma(x).$$

For a function f defined on $X \times (0, \infty)$, we define an area integral $A_\alpha(f)$ by

$$A_\alpha(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^2 \frac{d\mu(y)dt}{t^{\alpha+1}} \right)^{1/2}, \quad \alpha \in \mathbb{R} \tag{1.1}$$

for $x \in X$.

We then also define the *tent space* $T_2^p(X \times (0, \infty))$, $0 < p < \infty$, by

$$T_2^p(X \times (0,\infty)) = \{f : A_\alpha(f) \in L^p(d\mu)\}$$

with

$$||f||_{T_2^p} = ||A_\alpha(f)||_{L^p(d\mu)}.$$

For $0 < p \leq 1$, a function a, supported in $\hat{B}$ for some ball B in X, is said to be a $(p, 2)$-*atom* if

$$\int_{\hat{B}} |a(x,t)|^2 \frac{d\mu(x)dt}{t} \leq \mu(B)^{1-2/p}.$$

We now define certain generalized area integrals associated with appropriate singular measures on X. Let ν be a positive measure on X, and assume there exists a constant C so that

$$\nu(B(x,\rho)) \leq C\rho^\beta \tag{1.2}$$

for some fixed $\beta > 0$. Then, for fixed p, $0 < p \leq 1$, and a function f defined on $X \times (0, \infty)$, we define another area integral $G_{p,\alpha}(f)$ by

$$(1.3) \quad G_{p,\alpha}(f)(x) = \left(\int_{\Gamma(x)} |f(y,t)|^2 \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt \right)^{1/2}, \quad \alpha \in \mathbb{R}$$

for $x \in X$. Note that when $\alpha = \beta$ the two definitions (1.1) and (1.3) coincide.

In this paper we are concerned with the inequality for the L^p norms of area integrals $A_\alpha(f)$ and $G_{p,\alpha}(f)$ in $X \times (0, \infty)$; more precisely, there exists a constant C_p so that if $f \in T_2^p(X \times (0, \infty))$, $0 < p \leq 1$, and ν is a positive measure on X satisfying (1.2), then we have

$$||G_{p,\alpha}(f)||_{L^p(d\nu)} \leq C_p ||A_\alpha(f)||_{L^p(d\mu)}.$$

2. Main result

We state the two lemmas we need.

LEMMA 1. *Let (X, d, μ) be a space of homogeneous type. If $f \in T_2^p(X \times (0, \infty))$, $0 < p \leq 1$, then*

$$(2.1) \qquad |f(x,t)| \leq \sum_{j=0}^{\infty} \lambda_j a_j(x,t),$$

where the a_j's are $(p, 2)$-atoms, and the λ_j's are positive numbers. Moreover,

$$(2.2) \qquad \sum_{j=0}^{\infty} \lambda_j^p \leq C_p ||A_\alpha(f)||^p_{L^p(d\mu)}.$$

Proof. See Suh[3].

LEMMA 2. *Let (X, d, μ) be a space of homogeneous type, and let ν be a positive measure on X with the property (1.2). Suppose $0 < p \le 1$. Then there exists a constant C_p so that if a is a $(p, 2)$-atom supported in the tent $\hat{B}$ over a ball B having radius ρ, then*

$$\int_X [G_{p,\alpha}(a)(x)]^p d\nu(x) \le C_p.$$

Proof. Let a be a $(p, 2)$-atom supported in the tent $\hat{B}$ over a ball B having radius ρ, and $\chi_{B(y,t)}$ be the characteristic function of the ball $B(y, t)$. Then

(2.3)

$$\begin{aligned}
&\int_X [G_{p,\alpha}(a)(x)]^2 d\nu(x) \\
&= \int_X \left(\int_{\Gamma(x)} |a(y,t)|^2 \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt \right) d\nu(x) \\
&= \int_X \left(\int_{X\times(0,\infty)} |a(y,t)|^2 \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} \chi_{B(y,t)}(x) d\mu(y)dt \right) d\nu(x) \\
&\le C \int_{X\times(0,\infty)} |a(y,t)|^2 \frac{t^{\beta+2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt,
\end{aligned}$$

since

$$\int_X \chi_{B(y,t)}(x) d\nu(x) \le Ct^\beta.$$

Now observe that $t \le 2\rho$ for $(y, t) \in \hat{B}$ and small $\rho > 0$. So the last side of (2.3) is less than

$$\begin{aligned}
&C\rho^{(\alpha-\beta)(2/p-1)} \int_{\hat{B}} |a(y,t)|^2 \frac{d\mu(y)dt}{t} \\
\le &C\nu(B)^{1-2/p} \qquad (\text{by } 1 - 2/p < 0).
\end{aligned}$$

Thus

$$\int_X [G_{p,\alpha}(a)(x)]^2 d\nu(x) \le C\nu(B)^{1-2/p} \tag{2.4}$$

for some constant C. For $0 < p \leq 1$, Hölder's inequality and (2.4) give that

$$\int_X [G_{p,\alpha}(a)(x)]^p d\nu(x)$$
$$\leq \left(\int_X [[G_{p,\alpha}(a)(x)]^p]^{2/p} d\nu(x)\right)^{p/2} \left(\int_X [\chi_B(x)]^{2/(2-p)} d\nu(x)\right)^{(2-p)/2}$$
$$\leq \left(\int_X [G_{p,\alpha}(a)(x)]^2 d\nu(x)\right)^{p/2} \left(\int_X [\chi_B(x)]^{2/(2-p)} d\nu(x)\right)^{(2-p)/2}$$
$$\leq C_p$$

for some constant C_p. The proof is therefore complete.

The main result of this paper is now the following.

THEOREM 3. *Let (X, d, μ) be a space of homogeneous type, and let ν be a positive measure on X with the property (1.2). Then there exists a constant C_p so that if $f \in T_2^p(X \times (0, \infty))$, $0 < p \leq 1$, then*

$$||G_{p,\alpha}(f)||_{L^p(d\nu)} \leq C_p ||A_\alpha(f)||_{L^p(d\mu)}.$$

Proof. Let $f \in T_2^p(X \times (0, \infty))$, $0 < p \leq 1$ and write

$$|f(y,t)| \leq \sum_{j=0}^{\infty} \lambda_j a_j(y,t),$$

as in (2.1) of Lemma 1. Then we obtain

$$[G_{p,\alpha}(f)(x)]^2 \leq \int_{\Gamma(x)} [\sum_{j=0}^{\infty} \lambda_j a_j(y,t)]^2 \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt$$
$$= \sum_{i,j} \int_{\Gamma(x)} \lambda_i \lambda_j a_i(y,t) a_j(y,t) \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt$$
$$\leq \sum_{i,j} \left(\int_{\Gamma(x)} [\lambda_i a_i(y,t)]^2 \frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}} d\mu(y)dt\right)^{1/2}$$

$$\times\left(\int_{\Gamma(x)}[\lambda_j a_j(y,t)]^2\frac{t^{2(\alpha-\beta)/p}}{t^{\alpha+1}}d\mu(y)dt\right)^{1/2}$$
$$=\sum_{i,j}\lambda_i\lambda_j G_{p,\alpha}(a_i)(x)G_{p,\alpha}(a_j)(x)$$
$$=[\sum_{j=0}^{\infty}\lambda_j G_{p,\alpha}(a_j)(x)]^2.$$

Thus we obtain

$$[G_{p,\alpha}(f)(x)]^p\le[\sum_{j=0}^{\infty}\lambda_j G_{p,\alpha}(a_j)(x)]^p. \tag{2.5}$$

Integrate both sides of (2.5) with respect to $d\nu(x)$. Then it follows from (2.2) and Lemma 2 that

$$\begin{aligned}\int_X[G_{p,\alpha}(f)(x)]^p d\nu(x) &\le \int_X\sum_{j=0}^{\infty}\lambda_j^p[G_{p,\alpha}(a_j)(x)]^p d\nu(x)\\ &\le \sum_{j=0}^{\infty}\lambda_j^p\int_X[G_{p,\alpha}(a_j)(x)]^p d\nu(x)\\ &\le C_p\sum_{j=0}^{\infty}\lambda_j^p\\ &\le C_p||A_\alpha(f)||^p_{L^p(d\mu)}.\end{aligned}$$

Thus

$$||G_{p,\alpha}(f)||_{L^p(d\nu)}\le C_p||A_\alpha(f)||_{L^p(d\mu)}.$$

The proof is therefore complete.

References

[1] R. R. Coifman, Y. Meyer and E. M. Stein, *Some new function spaces and their applications to harmonic analysis*, J. Func. Anal. **62** (1985), 304–355

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