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CONTROL PROBLEMS OF INTEGRODIFFERENTIAL SYSTEMS IN BANACH SPACE

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1. Introduction

Several authors have studied the problem of controllability of linear and nonlinear systems in Banach spaces[2] Lasiecka and Triggiani[5] have studied exact controllability of abstract semilinear equations. Quinn and Carmichael[7] have shown that the controllability problem in Banach spaces can be converted into one of fixed point problems for a single valued mapping. Kwun et al.[4] investigated the approximate controllability and controllability of delay Volterra systems by using a fixed point theorem. Recently, Balanchandran et al. [1] studied the controllability of nonlinear integrodifferential systems in Banach spaces. In this paper, we shall study the controllability of semilinear integrodifferential systems in a Banach space by using the Schauder fixed point theorem.

2. Preliminaries

We consider the semilinear integrodifferential system:

$$
\begin{aligned}
(1)\qquad x'(t) &= A[x(t) + \int_0^t F(t-s)x(s)ds] + Bu(t) + f(t, x(t)) \\
&\quad + \int_0^t [a(t,s)g(s,x(s)) + h(t,s,x(s))]ds, \quad t \in [0,T], \\
x(0) &= x_0,
\end{aligned}
$$

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where the state $x(\cdot)$ takes values in the Banach space X and the control function $u(\cdot)$ is given in $L^2(J;U)$, a Banach space of admissible control functions with U as a Banach space. Here, A is a generator of a strongly continuous semigroup and B is a bounded linear operator from U into X. The nonlinear functions $f : J \times X \to X$, $g : J \times X \to X$, $h : J \times J \times X \to X$ and the kernel $a : J \times J \to \mathbb{R}$ ($\mathbb{R}$ denotes the set of real numbers) are continuous. Moreover, $F(t) \in B(X)$, $t \in J$, $F(t) : Y \to Y$ and for a continuous $x(\cdot)$ in Y, $AF(\cdot)X(\cdot) \in L^1([0,T];X)$. For $x \in X$, $F'(t)x$ is continuous in $t \in [0,T]$, where $B(X)$ is the space of all linear and bounded operators on X and Y is the Banach space formed from $D(A)$, the domain of A, endowed with the graph norm. Then for the system (1), there exists a mild solution of the following form :

$$
\begin{aligned}
x(t) &= R(t)x_0 + \int_0^t R(t-s)[Bu(s) + f(s,x(s)) \\
&\quad + \int_0^s (a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau)))d\tau]ds, \quad t \in [0,T], \\
x(0) &= x_0,
\end{aligned} \tag{2}
$$

where the resolvent operator $R(t) \in B(X)$ for $t \in J$ satisfies the following conditions:

(a) $R(0) = I$ (the identity operator on X).
(b) for all $x \in X$, $R(t)x$ is continuous for $t \in J$.
(c) $R(t) \in B(Y)$, $t \in J$. For $y \in Y$, $R(t)y \in C^1([0,T];X) \cap C^1([0,T];Y)$ and

$$
\begin{aligned}
\frac{d}{dt}R(t)y &= A[R(t)y + \int_0^t F(t-s)R(s)yds] \\
&= R(t)Ay + \int_0^t R(t-s)AF(s)yds], \quad t \in J.
\end{aligned}
$$

DEFINITION. The system (1) is said to be controllable on the interval J if for every x_0, $x_1 \in X$, there exists a control $u \in L^2(J;U)$ such that the solution $x(\cdot)$ of (1) satisfies $x(T) = x_1$.

We assume the following hypotheses:

(i) the resolvent operator $R(t)$ is compact such that $\max_{t>0} \|R(t)\| \leq M_1$.

(*ii*) the linear operator from U into X, defined by

$$Wu = \int_0^T R(t-s)Bu(s)ds$$

has an invertible operator W^{-1} defined on $L^2(J;U)/kerW$ and there exist positive constants M_2, M_3 such that $\|B\| \leq M_2$ and $\|W^{-1}\| \leq M_3$.

(*iii*) the nonlinear operators $f(t,x(t))$, $g(t,x(t))$, $h(t,s,x(s))$ and the kernel $a(t,s)$ for t, $s \in J$ satisfy

$$\begin{aligned}
&\|f(t,x(t)\| \leq M_4 \\
&\|g(t,x(t)\| \leq M_6 \\
&\|h(t,s,x(s)\| \leq M_7 \\
&\|a(t,s)\| \leq M_5, \quad \text{where } M_i > 0, \quad \text{for } i = 1,2,\cdots 7.
\end{aligned}$$

3. Main result

THEOREM 3.1. *If the hypotheses (i) – (iii) are satisfied, then the system (1) is controllable on J.*

Proof. Using the hypothesis (*ii*), an arbitrary function $x(\cdot)$ defines the control

$$u(t) = W^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s)\{f(s,x(s)) + \int_0^s (a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau)))d\tau\}ds], \quad t \in [0,T].$$

Now we shall show that when using this control, the operator defined by

$$(\Phi x)(t) = R(t)x_0 + \int_0^t R(t-s)[Bu(s) + f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds$$

has a fixed point. This fixed point is then a solution of equation (2). Clearly, $(\Phi x)(T) = x_1$, which means that the control u states the semilinear integrodifferential system from the initial state x_0 to x_1 in time T, provided we can obtain a fixed point of the nonlinear operator Φ. Let $Z = C(J; X)$ and

$$Z_0 = \{x \in Z : x(0) = x_0, \|\dot{x}(t)\| \le r \quad \text{for } t \in J\},$$

where the positive constant r is given by

$$\begin{aligned} r = & M_1\|x_0\| + M_1M_2M_3[\|x_1\| + M_1\|x_0\| + M_1\{M_4 \\ & + (M_5M_6 + M_7)T\}T]T + M_1[M_4 + (M_5M_6 + M_7)T]T. \end{aligned}$$

Then Z_0 is clearly a bounded, closed, convex subset of Z. Define a mapping $\Phi : Z \to Z_0$ by

$$\begin{aligned} (\Phi x)(t) = & R(t)x_0 + \int_0^t R(t-\eta)BW^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s) \\ & \times [f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds](\eta)d\eta \\ & + \int_0^t R(t-s)[f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds. \end{aligned}$$

We claim that $\Phi : Z_0 \to Z_0$ is continuous. From the hypothesis (iii), we get

$$\begin{aligned} \|(\Phi x)(t)\| \le & \|R(t)x_0\| + \int_0^t \|R(t-\eta)BW^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s) \\ & \times [f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds](\eta)\|d\eta \\ & + \int_0^t \|R(t-s)[f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau\|ds \\ \le & M_1\|x_0\| + M_1M_2M_3[\|x_1\| + M_1\|x_0\| + M_1\{M_4 \\ & + (M_5M_6 + M_7)T\}T]T + M_1[M_4 + (M_5M_6 + M_7)T]T \\ = & r. \end{aligned}$$

Since f, a, g and h are continuous and $\|(\Phi x)(t)\| \le r$, it follows that Φ is continuous and maps Z_0 into itself. Moreover, Φ maps Z_0 into a precompact subset of Z_0. To prove this, we first show that for every fixed $t \in J$, the set $Z_0(t) = \{(\Phi x)(t); x \in Z_0\}$ is precompact in X. This is clear for $t = 0$, since $Z_0(0) = \{x_0\}$. Let $t > 0$ be fixed and for $0 < \epsilon < t$, define

$$(\Phi_\epsilon x)(t) = R(t)x_0 + \int_0^{t-\epsilon} R(t-\eta)BW^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s)$$
$$\times [f(s, x(s)) + \int_0^s \{a(s,\tau)g(\tau, x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds](\eta)d\eta$$
$$+ \int_0^{t-\epsilon} R(t-s)[f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds.$$

Since $R(t)$ is compact for every $t > 0$, the set $Z_\epsilon(t) = \{(\Phi_\epsilon x)(t) : x \in Z_0\}$ is precompact in X for every ϵ, $0 < \epsilon < t$ Furthermore, for $x \in Z_0$, we have

$$\|(\Phi x)(t) - (\Phi_\epsilon x)(t)\| = \| \int_{t-\epsilon}^{t} R(t-\eta)BW^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s)$$
$$\times \{f(s,x(s)) + \int_0^s (a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau)))d\tau\}ds](\eta)d\eta\|$$
$$+ \| \int_{t-\epsilon}^{t} R(t-s)[f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds\|$$
$$\leq \epsilon M_1M_2M_3[\|x_1\| + M_1\|x_0\| + M_1\{M_4 + (M_5M_6 + M_7)T\}T]T$$
$$+ \epsilon M_1[M_4 + (M_5M_6 + M_7)T]T$$

which implies that $Z_0(t)$ is totally bounded, that is, precompact in X. We want to show that $\Phi(Z_0) = \{\Phi x : x \in Z_0\}$ is an equicontinuous family of functions. For that, let $t_2 > t_1 > 0$ Then we have

$$\|(\Phi x)(t_1) - (\Phi x)(t_2)\| \leq \|R(t_1) - R(t_2)\|\|x_0\| + \| \int_0^{t_1} [R(t_1 - \eta)$$
$$- R(t_2 - \eta)]BW^{-1}[x_1 - R(T)x_0 - \int_0^T R(t-s)\{f(s,x(s)) + \int_0^s (a(s,\tau)g(\tau, x$$
$$+ h(s,\tau,x(\tau)))d\tau\}ds](\eta)d\eta\| - \| \int_{t_1}^{t_2} R(t_2-\eta)BW^{-1}[x_1 - R(T)x_0$$

(3)

$$
\begin{aligned}
&- \int_0^T R(t-s)\{f(s,x(s)) + \int_0^s (a(s,\tau)g(\tau,x(\tau)) \\
&+ h(s,\tau,x(\tau)))d\tau\}ds](\eta)d\eta\| + \| \int_0^{t_1} [R(t_1-s) - R(t_2-s)]\{f(s,x(s)) \\
&+ \int_0^s (a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau)))d\tau\}ds - \int_{t_1}^{t_2} R(t_2-s) \\
&\times [f(s,x(s)) + \int_0^s \{a(s,\tau)g(\tau,x(\tau)) + h(s,\tau,x(\tau))\}d\tau]ds\| \\
\leq\ & \|R(t_1) - R(t_2)\| \|x_0\| + \int_0^{t_1} \|R(t_1-s) - R(t_2-s)\| M_2M_3[\|x_1\| \\
&+ M_1\|x_0\| + M_1M_4 + (M_5M_6 + M_7)TT]ds \\
&+ \int_{t_1}^{t_2} \|R(t_2-s)\| M_2M_3[\|x_1\| + M_1\|x_0\| + M_1\{M_4 + (M_5M_6 \\
&+ M_7)T\}T]ds + \int_0^{t_1} \|R(t_1-s) - R(t_2-s)\|\ [M_4 + (M_5M_6 \\
&+ M_7)T]ds + \int_{t_1}^{t_2} \|R(t_2-s)\| [M_4 + (M_5M_6 + M_7)T\}ds.
\end{aligned}
$$

The compactness of $R(t)$, $t > 0$ implies that $R(t)$ is continuous in the uniform operator topology for $t > 0$. Thus the right hand side of (3) which is independent of $x \in Z_0$, tends to zero as $t_1 - t_2 \to 0$. So $\Phi(Z_0)$ is an equicontinuous family of functions. Also, $\Phi(Z_0)$ is bounded in Z and so by the Arzela-Ascoli theorem, $\Phi(Z_0)$ is precompact. Hence from the Schauder fixed point theorem, Φ has a fixed point in Z_0. Any fixed point of Φ is a mild solution of the system (1) on J satisfying $(\Phi x)(t) = x(t) \in X$. Thus the system (1) is controllable on J.

EXAMPLE. We consider the following equations of the form:

$$
\begin{aligned}
y_t(t,x) &= \frac{\partial^2}{\partial x^2}[y(t,x) + \int_0^t F(t-s)y(s,x)ds] \\
&+ Bu(t) + f(t, y_{xx}(t,x)) \qquad (4) \\
&+ \int_0^t [a(t,s)g(t,s,y_{xx}(s,x)) + h(t,s,y_{xx}(s,x))]ds,
\end{aligned}
$$

and given initial and boundary conditions

$$y(0,t) = y(1,t) = 0, \quad x \in I = (0,1), \quad t \in J,$$
$$y(x,0) = y_0(x),$$

where $F : J \to \mathbb{R}$ is continuous and bounded, $B : U \to X$ with $U \subset J$ and $X = L^2(I;\mathbb{R})$ is linear operator such that there exists an invertible operator W^{-1} on $L^2(J;U)/kerW$, hence W is defined by

$$Wu = \int_0^T R(t-s)Bu(s)ds,$$

$R(t)$ is compact and

$$a; J \times J \to \mathbb{R},$$
$$f; J \times X \to X,$$
$$g, J \times X \to X,$$
$$h; J \times J \times X \to X$$

are all continuous and uniformly bounded. Let $X = L^2(I;\mathbb{R})$, $Ax = w_{xx}$, $B : U \to X$ and $D(A) = \{w \in X : w_{xx} \in X, w(0) = w(1) = 0\}$ be such that the condition in the hypothesis (ii) is satisfied. Then the system (4) becomes an abstract formulation of (1). Also the conditions of Theorem 3.1 are satisfied. Hence the system (4) is controllable on J.

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