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CERTAIN CLASSES OF MEROMORPHIC FUNCTIONS WITH POSITIVE COEFFICIENTS

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1. Introduction

Let Σ_p denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (a_n \geq 0) \tag{1.1}$$

which are analytic in $\mathcal{D} = \mathcal{U} - \{0\}$, where $\mathcal{U} = \{z : |z| < 1\}$. Let $\Sigma_p^*(\alpha)$ and $\Sigma_k(\alpha)(0 \leq \alpha < 1)$ denote the subclasses of Σ_p that are meromorphically starlike of order α and meromorphically convex of order α, respectively. Analytically, a function f of the form (1.1) is in $\Sigma_p^*(\alpha)$ if and only if

$$-Re\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha \ (z \in U). \tag{1.2}$$

Similarly, a function $f \in \Sigma_k(\alpha)$ if and only if f is of the form (1.1) and satisfies

$$-Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha \ (z \in U). \tag{1.3}$$

The class $\Sigma_p^*(\alpha)$ and related other classes have been extensively studied by Clunie[1], Libera[2], Pommerenke[4] and others.

Assume that $\{c_n\}_{n=0}^{\infty}$ is a sequence of positive real numbers such that the series $\Sigma_{n=1}^{\infty} c_n a_n z^n$ is absolutely convergent for every $z \in \mathcal{U}$ Moreover, we suppose that $c_0 \leq c_n (n \in \mathbb{N} = \{1, 2, \cdots\})$.

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Let $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ be the class of functions that consists of all functions f belonging to Σ_p such that

$$-\mathrm{Re}\left\{\frac{L(f(z))}{c_0 f(z)}\right\} > \alpha \ (0 \le \alpha < 1, \ z \in \mathcal{U}),$$

where

$$L(f(z)) = -\frac{c_0}{z} + \Sigma_{n=1}^\infty c_n a_n z^n \ (z \in \mathcal{D}).$$

In particular, if $c_0 = 1$ and $c_n = n (n \in \mathbb{N})$, then the class $\Sigma_p^\alpha(\{c_n)\}_{n=0}^\infty)$ reduces to the class $\Sigma_p^*(\alpha)$ studied by Mogra, Reddy and Juneja[3]. In the present paper, we prove coefficient estimates, distortion theorems, and convexity and starlikeness properties for the elements of $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$. Furthermore, modified Hadamard(or convolution) products in $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ are investigated.

2. Coefficient Estimates

THEOREM 2.1. *Let f be in the class Σ_p. If f is given by* (1.1), then $f \in \Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ if and only if

$$\sum_{n=1}^\infty (\alpha c_0 + c_n) a_n \le (1-\alpha) c_0. \tag{2.1}$$

Moreover, the result (2.1) is sharp, since the equality holds true for the function f given by

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_m} z^m \ (m \in \mathbb{N}, \ z \in \mathcal{D}). \tag{2.2}$$

Proof. Assume firstly that $f \in \Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$. Then

$$-\mathrm{Re}\left\{\frac{L(f(z))}{c_0 f(z)}\right\} = -\mathrm{Re}\left\{\frac{-c_0 + \sum_{n=1}^\infty c_n a_n z^{n+1}}{c_0(1 + \sum_{n=1}^\infty a_n z^{n+1})}\right\} > \alpha \tag{2.3}$$
$$(0 \le \alpha < 1, \ z \in \mathcal{U}).$$

Choose the values of z on the real axis so that $L(f(z))/c_0 f(z)$ is real. Upon clearing the denominator in (2.3) and letting $z \to 1^-$ through real values, we get

$$c_0 - \sum_{n=1}^{\infty} c_n a_n \geq \alpha c_0 \left(1 + \sum_{n=1}^{\infty} a_n\right)$$

or, equivalently,

$$\sum_{n=1}^{\infty} (\alpha c_0 + c_n) a_n \ \leq \ (1-\alpha) c_0.$$

We now prove that

$$\left|\frac{L(f(z))}{c_0 f(z)} + 1\right| < \left|\frac{Lf(z))}{c_0 f(z)} + 2\alpha - 1\right| \tag{2.4}$$

$$(0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ z \in \mathcal{U}),$$

provided that the condition (2.1) is satisfied. Note that (2.1) and (2.4) imply that $f \neq 0$ in $\mathcal{D}$ and $f \in \Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$, respectively Then we have

$$|z|(|L(f(z)) + c_0 f(z)| - |L(f(z)) + (2\alpha - 1)c_0 f(z)|)$$

$$= |z| \left(\left|\sum_{n=1}^{\infty} (c_0 + c_n) a_n z^n\right| - \left|2(\alpha - 1)c_0 \frac{1}{z} + \sum_{n=1}^{\infty} [(2\alpha - 1)c_0 + c_n] a_n z^n\right| \right)$$

$$\leq \sum_{n=1}^{\infty} (c_0 + c_n) a_n |z|^{n+1} - 2(1-\alpha)c_0 + \sum_{n=1}^{\infty} [(2\alpha - 1)c_0 + c_n] a_n |z|^{n+1}$$

$$= \sum_{n=1}^{\infty} (2\alpha c_0 + 2c_n) a_n |z|^{n+1} - 2(1-\alpha) c_0.$$

By letting $|z| \to 1^-$, we get

$$\sum_{n=1}^{\infty} (\alpha c_0 + c_n) a_n - (1-\alpha) c_0 \leq 0,$$

by (2.1). Therefore we conclude that f belongs to the class $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$. It is clear that the equality (2.1) holds true for the function given by (2.2), which evidently completes the proof of Theorem 2.1.

From Theorem 2.1, we have the following results immediately.

COROLLARY 2.1. *If a function f of the form* (1.1) belongs to the class $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$, then

$$a_n \le \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} \quad (n \in \mathbb{N}).$$

COROLLARY 2.2. *The class $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ is a convex subset of Σ_p.*

3. Distortion Theorems

Before proving some distortion properties for functions belonging to the class $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$, we need to establish the following result.

LEMMA 3.1. *A function f is in $\Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ if and only if there exist $d_n \ge 0 (n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\})$ such that $\sum_{n=0}^\infty d_n = 1$ and $f(z) = \sum_{n=0}^\infty d_n f_n(z) (z \in \mathbb{D})$, where*

$$f_0(z) = \frac{1}{z} \tag{3.1}$$

and

$$f_n(z) = \frac{1}{z} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} z^n \quad (n \in \mathbb{N}). \tag{3.2}$$

Proof. Let $f \in \Sigma_p^\alpha(\{c_n\}_{n=0}^\infty)$ be given by (1.1). Define

$$d_n = \frac{\alpha c_0 + c_n}{(1-\alpha)c_0} a_n \quad (n \in \mathbb{N})$$

and $d_0 = 1 - \sum_{n=1}^\infty d_n$. It is obvious that $d_n \ge 0 (n \in \mathbb{N}_0)$ and $\sum_{n=0}^\infty d_n = 1$. Moreover, we have

$$\begin{aligned} f(z) &= \frac{1}{z} + \sum_{n=1}^\infty a_n z^n \\ &= \frac{1}{z} + \sum_{n=1}^\infty d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} z^n \\ &= d_0 \frac{1}{z} + \sum_{n=1}^\infty d_n \left(\frac{1}{z} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} z^n \right) \quad (z \in \mathcal{D}). \end{aligned}$$

Conversely, if $f(z) = \sum_{n=0}^{\infty} d_n f_n(z) (z \in \mathcal{D})$, where $d_n \geq 0 (n \in \mathbb{N}_0)$ and $\sum_{n=0}^{\infty} d_n = 1$, then

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} z^n \ \ (z \in \mathcal{D}),$$

where

$$d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} \geq 0 \ \ (n \in \mathbb{N}).$$

Also, we have

$$\sum_{n=1}^{\infty} \frac{\alpha c_0 + c_n}{1-\alpha)c_0} \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} d_n = \sum_{n=1}^{\infty} d_n = 1 - d_0 \leq 1.$$

Hence, by virtue of Theorem 2.1, we conclude that $f \in \Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$.

We now obtain distortion results for functions belonging to $\Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$.

THEOREM 3.1. *Let f be in the class $\Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$.* (a) If $c_n \geq c_1 (n \in \mathbb{N})$, then

$$\frac{1}{|z|} - \frac{(1-\alpha)c_0}{\alpha c_0 + c_1}|z| \leq |f(z)| \leq \frac{1}{|z|} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_1}|z| \ \ (z \in \mathcal{D}). \tag{3.3}$$

(b) If $\{n/(\alpha c_0 + c_n)\}_{n=1}^{\infty}$ is a decreasing sequence, then

$$\frac{1}{|z|^2} - \frac{(1-\alpha)c_0}{\alpha c_0 + c_1} \leq |f'(z)| \leq \frac{1}{|z|^2} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_1} \ \ (z \in \mathcal{D}). \tag{3.4}$$

Equalities holds true in (3.3) and (3.4) for the function

$$f(z) = \frac{1}{z} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_1} z \ \ (0 \leq \alpha < 1, \ z \in \mathcal{D} \cap (0, \infty)). \tag{3.5}$$

Proof. Let $f \in \Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$. According to Lemma 3.1, we can write

$$f(z) = \sum_{n=0}^{\infty} d_n f_n(z) \ \ (z \in \mathcal{D}), \tag{3.6}$$

where $d_n \geq 0 (n \in \mathbb{N}_0)$ with $\sum_{n=0}^{\infty} d_n = 1$, and $f_n (n \in \mathbb{N}_0)$ are given by (3.1) and (3.2). For every $n \in \mathbb{N}$, we obtain

$$|f_n(z)| \leq \frac{1}{|z|} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_n}|z|^n \leq \frac{1}{|z|} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_1}|z| = f_1(|z|) \ (z \in \mathcal{D}),$$

because $c_n \geq c_1 (n \in \mathbb{N})$. Also, it is clear that $|f_0(z)| \leq f_1(|z|)(z \in \mathcal{D})$. Hence we have

$$|f(z)| \leq \sum_{n=0}^{\infty} d_n |f_n(z)| \leq f_1(|z|) = \frac{1}{|z|} + \frac{(1-\alpha)c_0}{\alpha c_0 + c_1}|z| \ (z \in \mathcal{D}).$$

Moreover, since $c_n \geq c_1 (n \in \mathbb{N})$, (3.6) and some well-known inequalities lead to

$$\begin{aligned} |f(z)| &\geq \frac{1}{|z|} - \left| \sum_{n=1}^{\infty} d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} z^n \right| \\ &\geq \frac{1}{|z|} - \sum_{n=1}^{\infty} d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n}|z|^n \\ &= \sum_{n=0}^{\infty} d_n \frac{1}{|z|} - \sum_{n=1}^{\infty} d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n}|z|^n \\ &= \sum_{n=1}^{\infty} d_n \left(\frac{1}{|z|} - \frac{(1-\alpha)c_0}{\alpha c_0 + c_n}|z|^n \right) + d_0 \frac{1}{|z|} \\ &\geq \frac{1}{|z|} - \frac{(1-\alpha)c_0}{\alpha c_0 + c_1}|z| \ (z \in \mathcal{D}), \end{aligned}$$

which establishes the second part of (3.3).

We now investigate the derivative $f'(z)(z \in \mathcal{D})$ of f. By differentiating term by term in (3.6), we get

$$f'(z) = \sum_{n=0}^{\infty} d_n f_n'(z) = -\frac{1}{z^2} + \sum_{n=1}^{\infty} d_n \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} n z^{n-1} \ (z \in \mathcal{D}).$$

Therefore we obtain

$$\frac{1}{|z|^2} - \max_{n \in \mathbb{N}} \left\{ \frac{(1-\alpha)c_0 n}{\alpha c_0 + c_n} \right\} \leq |f'(z)| \leq \frac{1}{|z|^2} + \max_{n \in \mathbb{N}} \left\{ \frac{(1-\alpha)c_0 n}{\alpha c_0 + c_n} \right\} \ (z \in \mathcal{D}).$$

The desired result (3.4) follows by taking into account the hypothesis that the sequence $\{n/(\alpha c_0 + c_n)\}_{n=1}^{\infty}$ is decreasing. Finally, it is not hard to see that the inequalities (3.3) and (3.4) are sharp.

4. Convexity and Starlikeness

We will investigate the radii of convexity and starlikeness of functions in $\Sigma_p^\alpha(\{c_n\}_{n=0}^{\infty})$ in theorems below.

THEOREM 4.1 *Let $f \in \Sigma_p^\alpha(\{c_n\}_{n=0}^{\infty})$ be given by (1.1). Then f is meromorphically convex of order δ in the disk $\{z \in \mathcal{U} : |z| < r\}$, where*

$$r = \inf_{n\in\mathbb{N}} \left\{ \frac{(1-\delta)(\alpha c_0 + c_n]}{(1-\alpha)c_0 n(n+2-\alpha)} \right\}^{\frac{1}{n+1}} \qquad (0 \leq \delta < 1),$$

provided that $r > 0$.

Proof. Let f be in $\Sigma_p^\alpha(\{c_n\}_{n=0}^{\infty})$. To see that f is meromorphically convex of order δ in the disk $\{z \in \mathcal{U} : |z| < r\}$, it is sufficient to prove that

$$\left| 2 + \frac{zf''(z)}{f'(z)} \right| \leq 1 - \delta, \tag{4.1}$$

provided that $|z| < r$. If f is given by (1.1), then

$$\begin{aligned} \left| 2 + \frac{zf''(z)}{f'(z)} \right| &= \left| \frac{f'(z) + (zf'(z))'}{f'(z)} \right| \\ &= \left| \frac{\sum_{n=1}^{\infty} n(n+1)a_n z^{n-1}}{-\frac{1}{z^2} + \sum_{n=1}^{\infty} n a_n z^{n-1}} \right| \\ &\leq \frac{\sum_{n=1}^{\infty} n(n+1)a_n |z|^{n+1}}{1 - \sum_{n=1}^{\infty} n a_n |z|^{n+1}}. \end{aligned}$$

Hence (4.1) is satisfied for $|z| < r$, provided that

$$\sum_{n=1}^{\infty} n(n+1)a_n|z|^{n+1} \leq (1 - \sum_{n=1}^{\infty} n a_n |z|^{n+1})(1-\delta) \quad (|z| < r, z \in \mathcal{U})$$

or, equivalently,

$$\sum_{n=1}^{\infty} n(n+2-\delta)a_n|z|^{n+1} \leq 1 - \delta \quad (0 \leq \delta < 1, |z| < r, z \in \mathcal{U}). \tag{4.2}$$

Finally, by virtue of Theorem 2.1, (4.2) holds true.

THEOREM 4.2. *Let $f \in \Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$. Then f is meromorphically starlike of order δ in the disk $\{z \in \mathcal{U} : |z| < r\}$, where*

$$r = \inf_{n \in \mathbb{N}} \left\{ \frac{(1-\delta)(\alpha c_0 + c_n]}{(1-\alpha)c_0(n+2-\delta)} \right\}^{\frac{1}{n+1}} \quad (0 \leq \delta < 1),$$

provided that $r > 0$.

Proof. Let f be in $\Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})$. It is sufficient to show that

$$\left|1 + \frac{zf'(z)}{f(z)}\right| \leq 1 - \delta, \tag{4.3}$$

provided that $|z| < r$. If f is given by (1.1), then we note that

$$\begin{aligned} \left|1 + \frac{zf'(z)}{f(z)}\right| &= \left|\frac{\sum_{n=1}^{\infty}(n+1)a_n z^n}{\frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n}\right| \\ &\leq \frac{\sum_{n=1}^{\infty}(n+1)a_n|z|^{n+1}}{1 - \sum_{n=1}^{\infty} a_n|z|^{n+1}} \\ &\leq 1 - \delta, \end{aligned}$$

and so

$$\sum_{n=1}^{\infty} (n+2-\delta)\, a_n|z|^{n+1} \leq 1 - \delta. \tag{4.4}$$

Therefore, from Theorem 2.1, (4.3) holds true.

5. Hadamard (or Convolution) Products

Let $f_i \in \Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})(i = 1, 2)$. If

$$f_i(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,i} z^n \quad (a_{n,i} \leq 0 (i = 1, 2), z \in \mathcal{D}), \tag{5.1}$$

we define the modified Hadamard(or convolution) product $f_1 * f_2$ of f_1 and f_2 by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_{n,1} a_{n,2} z^n \quad (z \in \mathcal{D}).$$

We now prove

THEOREM 5.1. *Let* $f_i \in \Sigma_p^{\alpha}(\{c_n\}_{n=0}^{\infty})(i = 1, 2)$. *If* $\{c_n\}_{n=0}^{\infty}$ *is an increasing sequence, then* $f_1 * f_2 \in \Sigma_p^{\gamma}(\{c_n\}_{n=0}^{\infty})$, *where*

$$\gamma = \frac{(\alpha c_0 + c_1)^2 - (1-\alpha)^2 c_0 c_1}{(1-\alpha)^2 c_0^2 + (\alpha c_0 + c_1)^2}$$

Proof. According to Theorem 2.1, if $f_i (i = 1,\ 2)$ is defined by (5.1), then

$$\sum_{n=1}^{\infty} \frac{\alpha c_0 + c_n}{(1-\alpha)c_0} a_{n,i} \leq 1 \quad (i = 1, 2). \tag{5.2}$$

We must show that

$$\sum_{n=1}^{\infty} \frac{\gamma c_0 + c_n}{(1-\gamma)c_0} a_{n,1} a_{n,2} \leq 1. \tag{5.3}$$

By virtue of the Cauchy-Schwarz inequality, (5.2) leads to

$$\sum_{n=1}^{\infty} \frac{(\alpha c_0 + c_n)}{(1-\alpha)c_0} \sqrt{a_{n,1} a_{n,2}} \leq 1. \tag{5.4}$$

Hence, in order to prove (5.3), it is sufficient to establish that

$$\frac{\gamma c_0 + c_n}{(1-\gamma)c_0)} a_{n,1} a_{n,2} \leq \frac{\alpha c_0 + c_n}{(1-\alpha)c_0} \sqrt{a_{n,1} a_{n,2}} \quad (n \in \mathbb{N}). \tag{5.5}$$

Moreover, from (5.4), we deduce that

$$\sqrt{a_{n,1} a_{n,2}} \leq \frac{(1-\alpha)c_0}{\alpha c_0 + c_n} \quad (n \in \mathbb{N}). \tag{5.6}$$

Therefore, by combining (5.4) and (5.6), we will obtain (5.5) when we have proved that

$$\frac{(1-\alpha)c_0}{\alpha c_0 + c_n} \leq \frac{1-\gamma}{1-\alpha} \frac{\alpha c_0 + c_n}{\gamma c_0 + c_n}$$

or, equivalently, that

$$\gamma \leq \frac{(\alpha c_0 + c_n)^2 - (1-\alpha)^2 c_0 c_n}{(1-\alpha)^2 c_0^2 + (\alpha c_0 + c_n)^2} \quad (n \in \mathbb{N}). \tag{5.7}$$

Inequality (5.7) is true because $\{c_n\}_{n=0}^{\infty}$ is an increasing sequence and the function

$$\zeta(x) = \frac{(\alpha c_0 + x)^2 - (1-\alpha)^2 c_0 x}{(1-\alpha)^2 c_0^2 + (\alpha c_0 + x)^2}$$

is increasing for positive real numbers x. This completes the proof of Theorem 5.1.

Remark. Taking $c_n = n (n \in \mathbb{N}_0)$ in Theorem 5.1, we have the result of Mogra, Reddy and Juneja[3].

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