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AUTOMORPHISMS OF METACYCLIC GROUPS OF PRIME POWER ORDER

Hyo-Seob Sim

1. Introduction

There are a number of determinations of the isomorphism types of finite metacyclic p-groups in the literature (see, for example, Beyl [1], King [2], Newman and Xu [3], and Sim [6]). When p is odd, the results in the literature typically take this form. given an odd prime p, each finite metacyclic p-group has *precisely* one presentation

$$\langle\, x, y \mid x^{p^\alpha} = y^{p^\beta},\ \ y^{p^\gamma} = 1,\ \ y^x = y^{1+p^\delta} \,\rangle$$

with the parameters $\alpha, \beta, \gamma, \delta$ subject to certain restrictions. (The choice of the restrictions varies from paper to paper, to fit the approach adopted)

Our main result presents a determination of the automorphism group of a finite nonabelian metacyclic p-group in terms of the parameters which are invariants of the isomorphism type, by using the presentations of the above form.

Let G be a metacyclic group and let K be a cyclic normal subgroup of G such that G/K is cyclic. Then the automorphism group of G acts on the set of all such cyclic normal subgroups. Our strategy to determine the automorphisms of a metacyclic p-group G may be stated as follows: we first investigate the orbit containing K under the action of the automorphism group of G, and then we try to determine all automorphisms of G that fix K setwise. This enables us to have the main results as follow.

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THEOREM 1.1. *Let P be a finite nonabelian split metacyclic p-group for an odd prime p.*

(i) P has the presentation of the form

$$P = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ a^b = a^{1+p^\gamma} \rangle,$$

where $0 < \gamma < \beta \leq \alpha + \gamma$, different values of parameters α, β, γ giving nonisomorphic such groups.

(ii) $AutP$ is a soluble group of order $(p-1)p^{\min(\alpha,\beta)+\min(\alpha,\gamma)+\alpha+\gamma-1}$ and each Hall p'-subgroup of $AutP$ is isomorphic to $\mathbb{Z}_{p-1}$.

THEOREM 1.2. *Let P be a finite nonsplit metacyclic p-group for an odd prime p.*

(i) P has the presentation of the form

$$P = \langle a, b \mid a^{p^\alpha} = b^{p^\beta},\ b^{p^{\beta+\delta}} = 1,\ b^a = b^{1+p^\gamma} \rangle$$

where $0 < \delta \leq \gamma < \beta < \alpha$, different values of parameters $\alpha, \beta, \gamma, \delta$ giving nonisomorphic such groups.

(ii) $AutP$ is a p-group of order $p^{\alpha+\beta+\gamma+\delta}$.

2. Background results

We first setup some notation and terminology.

Let m and n be positive integers. Define

$$|m \bmod n| := \min\{i \in \mathbb{Z} : i > 0,\ m^i \equiv 1 \bmod n\},$$

the multiplicative order of m modulo n; it is not defined unless $\gcd(m, n) = 1$.

We denote the commutator subgroup of a group G by G', the centre by $Z(G)$ and the Frattini subgroup by $\Phi(G)$. The automorphism group of a group G is denoted by $\mathrm{Aut}(G)$. If G is a finite p-group, then $\Omega_1(G)$ denotes the subgroup generated by all elements of order p. For two subgroups H and K of a group G, let $\mathrm{C}_H(K)$ denote the centralizer of K in H and let $\mathrm{N}_H(K)$ denote the normalizer of K in H.

We now collect some basic properties of metacyclic groups, which will be used later. The proofs of the results presented here can be found in [6] if the relevant references were not given. In this note we only deal with *finite* metacyclic groups, so for simplicity, by a metacyclic group we shall mean a finite metacyclic group.

Let G be a metacyclic group and let K be a cyclic normal subgroup of G such that G/K is cyclic. Then there exists a cyclic subgroup S such that $G = SK$. Such a factorization is called a metacyclic factorization. In particular, if G has a split metacyclic factorization, namely $G = SK$ such that $S \cap K = 1$, then G is called *split metacyclic*.

LEMMA 2.1. *Let G be a group with a metacyclic factorization $G = SK$. Let $S = \langle a \rangle$ and $K = \langle b \rangle$. Let r be an integer such that $a^{-1}ba = b^r$. Set $s := |r \bmod |K||$ and $t := |K|/\gcd(|K|, r-1)$. Then $G' = \langle a^{r-1} \rangle \cong \mathbb{Z}_t$, $Z(G) = \langle a^s, b^t \rangle$ and $S/C_S(K) \cong \mathbb{Z}_s$.*

LEMMA 2.2. *Let P be a metacyclic p-group for an odd prime p and let $P = SK$ be a metacyclic factorization. Then $S/C_S(K) \cong P'$.*

LEMMA 2.3. [7, Theorem 4.3.14] *Let P be a metacyclic p-group for an odd prime p. Then P is regular, and so $(ab^{-1})^{p^m} = 1 \iff a^{p^m} = b^{p^m}$, for every a, b in P and every nonnegative integer m. If $|P'|$ divides n, then $(a^ib^j)^n = a^{in}b^{jn}$.*

LEMMA 2.4. *If P is a noncyclic metacyclic p-group for an odd prime p,*

(i) $P/\Phi(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$;

(ii) $\Omega_1(P) \cong \mathbb{Z}_p \times \mathbb{Z}_p$.

LEMMA 2.5. *Let P be a noncyclic metacyclic p-group for an odd prime p and K a subgroup. Then*

(i) K is cyclic if and only if K does not contain $\Omega_1(P)$;

(ii) K is normal and P/K is cyclic if and only if K contains P' and K is not contained in $\Phi(P)$.

LEMMA 2.6. *Let P be a metacyclic p-group for an odd prime p. If K and P/K are cyclic, then there exists a cyclic subgroup S of P such that $P = SK$ and $|S| = \exp P$.*

We will also need the following lemma; a stronger version of it can be found in Rose [5, p. 193].

Lemma 2.7. *If C is a cyclic subgroup of a finite abelian group A such that $|C| = \exp A$, then C is a direct factor of A.*

3. Presentations of metacyclic p-groups

In this section, we consider canonical presentations for metacyclic p-groups for an odd p.

Let p be an odd prime and let P be a noncyclic metacyclic p-group. Let $P = SK$ be a metacyclic factorization of P. Define $\alpha, \beta, \gamma, \delta$ by

$$p^\alpha = |S : S \cap K|,\ p^\beta = |K : S \cap K|,\ p^\gamma = |K : P'|,\ p^\delta = |S \cap K|.$$

Then it is easy to show that P has the presentation

$$\langle\, a, b \mid a^{p^\alpha} = b^{p^\beta},\ \ b^{p^{\beta+\delta}} = 1,\ \ b^a = b^{1+p^\gamma} \,\rangle$$

and $0 \le \delta \le \gamma \le \beta + \delta \le \alpha + \gamma,\ 1 \le \gamma$. The presentation is obtained by choosing relevant generators a of S and b of K from the metacyclic factorization $P = SK$; we call this presentation a *metacyclic presentation* of P, and we denote it by $\wp(\alpha, \beta, \gamma, \delta)$. Note that the parameters of the presentation are not invariants of the isomorphism type of P in general, but determined by the choice of a metacyclic factorization of P.

If $\delta = 0$, then P is a split metacyclic; in this case, we denote the corresponding metacyclic presentation simply by $\wp(\alpha, \beta, \gamma)$ and call it a *split* metacyclic presentation of P. A metacyclic p-group is split if and only if it has a split metacyclic presentation.

Consider the set

$$\{K :\ K \trianglelefteq P,\ K \text{ and } P/K \text{ cyclic }\}.$$

The subgroups of minimal order in this set will be called *minimal kernels.* A metacyclic factorization $P = SK$ is called *standard* if $|S| = \exp P$ and K is a minimal kernel. Lemma 2.6 guarantees that P has a standard metacyclic factorization. The corresponding metacyclic presentation to a standard metacyclic factorization $P = SK$ is called a *standard* metacyclic presentation of P.

As a special case of the general observation for the metacyclic groups of odd order in [6], we see that a metacyclic presentation $\wp(\alpha,\beta,\gamma,\delta)$ of a noncyclic metacyclic p-group P is standard if and only if the parameters

$$0 \leq \delta \leq \gamma \leq \beta \leq \alpha,\ 1 \leq \gamma,$$

and so the parameters are invariants of the isomorphism type of P.

We now consider split noncyclic metacyclic p-groups. Let $\wp(\alpha,\beta,\gamma,\delta)$ be a standard metacyclic presentation of P. If $\delta = 0$, then P is obviously split; if $\alpha = \beta$, then $P = \langle ab \rangle \langle b \rangle$ is split; if $\beta = \gamma$, then $P = \langle ab^{p^{\alpha-\beta}} \rangle \langle a \rangle$ is split. Suppose now that P is split. Then P has a split metacyclic presentation $\wp(\alpha',\beta',\gamma')$, where $\alpha = \max\{\alpha',\gamma'\}$, $\beta = \min\{\alpha',\beta'\}$, $\gamma = \min\{\alpha',\gamma'\}$, $\delta = \max\{\alpha',\beta'\} - \min\{\alpha',\gamma'\}$. Therefore either $\alpha = \beta$, or $\beta = \gamma$, or $\delta = 0$. Consequently, P is split if and only if either $\alpha = \beta$, or $\beta = \gamma$, or $\delta = 0$ We also observe that $\wp(\alpha',\beta',\gamma')$ is the unique split metacylic presentation of P if P is nonabelian. This means the parameters for the split metacyclic presentation of P are invariants of the isomorphism type, provided P is not abelian.

By summarizing the above observation, we have

THEOREM 3.1. *Let p be an odd prime.*

(*i*) *Every finite nonabelian split metacyclic p-group P has a presentation of the form*

$$P = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ b^a = b^{1+p^\gamma} \rangle$$

where α,β,γ are positive integers such that $1 \leq \gamma < \beta \leq \alpha + \gamma$. Conversely, each such presentation defines a nonabelian split metacyclic p-group of order $p^{\alpha+\beta}$, different values of the parameters α,β,γ (with the above condition) giving nonisomorphic such groups.

(*ii*) *Every finite nonsplit metacyclic p-group P has a presentation of the form*

$$P = \langle a, b \mid a^{p^\alpha} = b^{p^\beta},\ b^{p^{\beta+\delta}} = 1,\ b^a = b^{1+p^\gamma} \rangle$$

where $\alpha,\beta,\gamma,\delta$ are nonnegative integers such that $1 < \delta \leq \gamma < \beta < \alpha$ Conversely, each such presentation defines such a metacyclic p-group of order $p^{\alpha+\beta+\delta}$, different values of the parameters $\alpha,\beta,\gamma,\delta$ (with the above condition) giving nonisomorphic groups.

4. Automorphisms of metacyclic p-groups

In this section, we determine the automorphism groups of nonabelian metacyclic p-groups for an odd prime p. With Theorem 3.1, this will complete the proofs of the main results.

We first observe the following lemma.

LEMMA 4.1. *For an odd prime p, let P be a finite nonabelian metacyclic p-group with a metacyclic factorization $P = SK$ such that P' properly contains $S \cap K$. There exists a one-to-one correspondence between the set $\{K\theta : \theta \in \mathrm{Aut}P\}$ and $\mathrm{Hom}(SP'/P', K/P')$, the set of all homomorphisms from SP'/P' to K/P'.*

Proof. Let θ be an automorphism of P. Since the cyclic p-group $K\theta$ contains P', and $S \cap K < P'$, we have $S \cap K\theta = S \cap K < P'$. Thus $P = S(K\theta)$ is a metacyclic factorization of P. It follows that $K\theta/P'$ is a direct complement of SP'/P' in P/P'.

On the other hand, let Y/P' be a direct complement of SP'/P' in P/P'. Then $SY = P$ and $SP' \cap Y = P'$. Since SP' contains $\Omega_1(P)$ and $SP' \cap Y = P'$ is cyclic, the subgroup Y does not contains $\Omega_1(P)$; so Y is cyclic by Lemma 2.5. Since $P' \leq Y$, we see that Y is also normal in P. Consequently, $P = SY$ is a metacyclic factorization such that $|K| = |Y|$ and $S \cap Y = S \cap K$. Therefore we see that the metacyclic factorizations $P = SK$ and $P = SY$ yield the same metacyclic presentation of P. This gives an automorphism θ of P such that $K\theta = Y$.

Consequently, we now have showed that there exists a one-to-one correspondence between $\{K\theta : \theta \in \mathrm{Aut}P\}$ and the set of all complements of SP'/P in P/P'. Since P/P' is the direct product of SP'/P' and K/P', the lemma is now clear from (11.1.2) in [4].

Let p be an odd prime and let P be a nonabelian metacyclic p-group. P has a presentation of the form

$$\langle a, b \mid a^{p^\alpha} = b^{p^\beta},\ \ b^{p^{\beta+\delta}} = 1,\ \ b^a = b^{1+p^\gamma} \rangle,$$

where $1 \leq \gamma,\ 0 \leq \delta \leq \gamma \leq \beta + \delta \leq \alpha + \gamma$. Let $S = \langle a \rangle$ and let $K = \langle b \rangle$. By Lemma 2.1 and Lemma 2.2, $P' = \langle b^{p^\gamma} \rangle$ and $C_P(K) = \langle a^{p^{\beta+\delta-\gamma}}, b \rangle$; so $|P'| = p^{\beta+\delta-\gamma}$. Since P is regular (see Lemma 2.3), $(a^i b^j)^{p^n} = a^{ip^n} b^{jp^n}$ for all integers i, j, n with $n \geq \beta + \delta - \gamma$.

Let
$$A(K) := \{x \in P : y^x = y^{1+p^\gamma}, y \in K\}.$$
Then $A(K) = a\mathrm{C}_P(K)$. Let N be the set of all automorphisms θ of P such that $K\theta = K$. Then N is a subgroup of $\mathrm{Aut}P$. The map

$$a \mapsto x,\ b \mapsto y$$

defines an automorphism in N if and only if $x \in A(K)$, $\langle y \rangle = K$ and $x^{p^\alpha} = y^{p^\beta}$. Define
$$\mathcal{K} := \{K\theta : \theta \in \mathrm{Aut}P\}$$
Then since $\mathrm{Aut}(P)$ acts transitively on $\mathcal{K}$, we have

$$|\mathrm{Aut}P| = |N||\mathcal{K}|.$$

Let r, s, t be integers such that

$$0 \le r < p^{\alpha-\beta+\gamma-\delta},\ 0 \le s < p^{\beta+\delta},\ 0 \le t < p^{\beta+\delta}.$$

The map
$$a \mapsto a^{1+rp^{\beta+\delta-\gamma}} b^{s-1},\ b \mapsto b^t$$
defines an automorphism in N if and only if

$$\gcd(p,t) = 1,\ tp^\beta \equiv p^\beta(1 + rp^{\beta+\delta-\gamma}) + sp^\alpha \bmod p^{\beta+\delta}.$$

In this case, the automorphism so defined is denoted by $\theta_{r,s,t}$. On the other hand, let θ be an automorphism in N. Then $a\theta \in a\mathrm{C}_P(K)$. Since $\mathrm{C}_P(K) = \langle a^{p^{\beta+\delta-\gamma}}, b\rangle$, we have $a\theta = a^{1+rp^{\beta+\delta-c}} b^s$ for some integers r, s such that $0 \le r < p^{\alpha-\beta+\gamma-\delta}$ and $0 \le s < p^{\beta+\delta}$. Obviously $b\theta = b^t$ for some integer t with $0 \le t < p^{\beta+\delta}$. Therefore, N consists of the automorphisms $\theta_{r,s,t}$ with the above conditions.

We now consider a nonabelian split metacyclic p-group P. By Theorem 3.1, P has the presentation as the above for some integers α, β, γ such that $1 \le \gamma < \beta \le \alpha + \gamma$. By the above observation, we have

$$\theta_{r,s,t} \in N \iff \gcd(p,t) = 1,\ sp^\alpha \equiv 0 \bmod p^\beta.$$

Thus $|N| = (p-1)p^{\alpha+\gamma-1+\min(\alpha,\beta)}$. On the other hand, observing that $|SP'/P'| = p^\alpha$ and $|K/P'| = p^\gamma$ we have, from Lemma 4.1, that $|\mathcal{K}| = p^{\min(\alpha,\gamma)}$ and so $|\mathrm{Aut}P| = (p-1)p^{\min(\alpha,\beta)+\min(\alpha,\gamma)+\alpha+\gamma-1}$. Moreover, each Hall p'-subgroup of Aut$P$ is isomorphic to the cyclic group of order $p-1$, and so Aut$P$ is soluble.

We finally consider a nonsplit metacyclic p-group P. By Theorem 3.1 again, we can choose the parameters in the above presentation for P so that $1 < \delta \leq \gamma < \beta < \alpha$. We note that P' contains $S \cap K$ properly. By applying Lemma 4.1, we get $|\mathcal{K}| = p^{\min(\alpha,\gamma)} = p^\gamma$ in this case. We also have

$$\theta_{r,s,t} \in N \iff \gcd(p,t) = 1,\ sp^{\alpha-\beta} \equiv t-1 \bmod p^\delta$$

from the above observation. There exist precisely $p^{2\beta+\delta}$ different pairs of integers s and t satisfying the condition. So we have $|\mathrm{Aut}P| = p^{\alpha+\beta+\gamma+\delta}$.

We summarize the observation as follow:

THEOREM 4.2. *For an odd prime p, let P be a finite nonabelian metacyclic p-group.*

(i) If P is presented by

$$P = \langle a, b \mid a^{p^\alpha} = 1,\ b^{p^\beta} = 1,\ a^b = a^{1+p^\gamma} \rangle,$$

where $0 < \gamma < \beta \leq \alpha + \gamma$, then

$$|AutP| = (p-1)p^{\min(\alpha,\beta)+\min(\alpha,\gamma)+\alpha+\gamma-1}.$$

Moreover, each Hall p'-subgroup of AutP is isomorphic to $\mathbb{Z}_{p-1}$, and so AutP is soluble.

(ii) If P is presented by

$$P = \langle a, b \mid a^{p^\alpha} = b^{p^\beta},\ b^{p^{\beta+\delta}} = 1,\ b^a = b^{1+p^\gamma} \rangle$$

where $\alpha, \beta, \gamma, \delta$ are nonnegative integers such that $1 < \delta \leq \gamma < \beta < \alpha$, then AutP is a p-group of order $p^{\alpha+\beta+\gamma+\delta}$.

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Division of Mathematical Sciences
Pukyong National University
Pusan 608-737, Korea