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A NOTE ON SAALSCHÜTZ'S THEOREM

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1. Introduction and Preliminaries

The generalized hypergeometric function with p numerator and q denominator parameter is defined by

$$
\begin{aligned}
(1.1)\qquad & {}_pF_q\left[\begin{matrix}\alpha_1\ \alpha_2\ \cdots\ \alpha_p; \\ \beta_1\ \beta_2\ \cdots\ \beta_q;\end{matrix}\ z\right] \\
&= {}_pF_q(\alpha_1\ \alpha_2\ \cdots\ \alpha_p;\ \beta_1\ \beta_2\ \cdots\ \beta_q; z) \\
&= \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ \cdots\ (\alpha_p)_n}{(\beta_1)_n\ \cdots\ (\beta_q)_n}\frac{z^n}{n!},
\end{aligned}
$$

where the Pochhammer symbol $(\alpha)_n$ (or the shifted factorial, since $(1)_n = n!$), is defined by, α any complex number,

$$
(1.2)\quad (\alpha)_n := \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n \in \mathbf{N} := \{1,\ 2,\ 3,\ \cdots\} \\ 1 & \text{if } n = 0, \end{cases}
$$

which, in view of the fundamental functional relation of the Gamma function Γ

$$
(1.3)\qquad \Gamma(\alpha+1) = \alpha\Gamma(\alpha) \quad \text{and} \quad \Gamma(1) = 1,
$$

is written in the equivalent form:

$$
(1.4)\qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},
$$

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where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$\{\Gamma(z)\}^{-1} = ze^{\gamma z} \prod_{k=1}^{\infty} \left\{ \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}} \right\}, \tag{1.5}$$

where γ is the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n \to \infty} \left(\sum_{k=1}^{n} \frac{1}{k} - \log n \right) \cong 0.577\,215\,664 \ldots . \tag{1.6}$$

From the definition (1.2), it is easy to see that

$$(\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k} \qquad (0 \leq k \leq n), \tag{1.7}$$

which, for $\alpha = 1$, reduces immediately to

$$(n-k)! = \frac{(-1)^k n!}{(-n)_k} \quad (0 \leq k \leq n;\ n \in \mathbf{N}). \tag{1.8}$$

The generalized binomial coefficient $\binom{\alpha}{n}$ is defined by

$$\binom{\alpha}{n} := \begin{cases} \dfrac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!} & \text{if } n \in \mathbf{N} \\ 1 & \text{if } n = 0, \end{cases} \tag{1.9}$$

which, in virtue of (1.2) and (1.3), is expressed in the following equivalent form:

$$\binom{\alpha}{n} = \frac{\Gamma(\alpha+1)}{n!\Gamma(\alpha-n+1)} = \frac{(-1)^n (-\alpha)_n}{n!}, \tag{1.10}$$

from which (1.2) may be extended to

$$(\alpha)_{-n} = \frac{\Gamma(\alpha-n)}{\Gamma(\alpha)} = \frac{(-1)^n}{(1-\alpha)_n} \quad (\alpha \neq 0,\ \pm 1,\ \pm 2,\ \cdots;\ n \in \mathbf{N}). \tag{1.11}$$

In this note we first recall a generalized form of the Saalschütz's theorem which is one of the important and useful theorems in ${}_3F_2$. We also investigate connections among some theorems associated with the generalized hypergeometric series.

2. Relations among Theorems

We start with a transformation formula due to Whipple [5, p. 263]

(2.1)

$$
{}_4F_3\left[\begin{matrix} t, x, y, z; \\ u, v, w, \end{matrix} 1\right]
= \frac{\Gamma(v+w-t)\Gamma(1+x-u)\Gamma(1+y-u)\Gamma(1+z-u)}{\Gamma(1+y+z-u)\Gamma(1+z+x-u)\Gamma(1+x+y-u)\Gamma(1-u)}
\times {}_7F_6\left[\begin{matrix} a, \quad 1+\frac{1}{2}a, \quad w-t, \quad v-t, \quad x, \quad y, \quad z; \\ \frac{1}{2}a, v, w, 1+y+z-u, 1+z+x-u, 1+x+y-u; \end{matrix} 1\right],
$$

which transforms a terminating well-poised ${}_7F_6$ into a series ${}_4F_3$ and $a = x+y+z-u$, $u+v+w = t+x+y+z+1$, while one of the four t, x, y, z is a negative integer.

Setting $v = t$ in (2.1) reduces immediately to Saalschüt's theorem:

(2.2)

$$
{}_3F_2\left[\begin{matrix} x, y, z; \\ u, w; \end{matrix} 1\right] = \frac{\Gamma(u)\Gamma(1-w+x)\Gamma(1-w+y)\Gamma(1-w+z)}{\Gamma(1-w)\Gamma(u-x)\Gamma(u-y)\Gamma(u-z)}
$$

provided that x, y or z is a negative integer and $u+w = x+y+z+1$.

By using (1.4), (2.2) is written in the following form

$$
{}_3F_2\left[\begin{matrix} -n, \quad a, \quad b; \\ c, 1-c+a+b-n; \end{matrix} 1\right] = \frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n} \quad (n \in \mathbf{N} \cup \{0\}). \tag{2.3}
$$

A useful well-known asymptotic formula for Gamma function is also provided:

$$
\frac{\Gamma(z+\alpha)}{\Gamma(z+\beta)} = z^{\alpha-\beta}\left[1 + O\left(\frac{1}{z}\right)\right] \quad (z \to \infty;\ |\arg z| < \pi). \tag{2.4}
$$

Taking the limit in (2.3) as $n \to \infty$ with the aid of (2.4) yields Gauss's summation formula

$$
{}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} \quad (c \neq 0, -1, -2, \cdots;\ \mathrm{Re}(c-a-b) > 0), \tag{2.5}
$$

which plays a vital role in the theory of hypergeometric series and can be deduced in several ways.

Replacing a or b by a nonpositive integer $-n$, we have a summation formula

$$(2.6)\quad {}_2F_1(-n, b; c; 1) = \frac{(c-b)_n}{(c)_n} \quad (n \in \mathbf{N}\cup\{0\};\ c \neq 0, -1, -2, \dots),$$

which incidentally can be shown to be equivalent to so-called Vandermonde's convolution theorem (see Choi [2, p. 159]; also Srivastava et al. [4, p. 19]):

$$(2.7)\qquad \sum_{k=0}^{n} \binom{\lambda}{k}\binom{\mu}{n-k} = \binom{\lambda+\mu}{n} \quad (n \in \mathbf{N}\cup\{0\}).$$

Indeed, using (1.8) and (1.10), the left side of (2.7), say S_n, becomes

$$\begin{aligned} S_n &= \sum_{k=0}^{n} \left[\frac{(-1)^k(-\lambda)_k}{k!}\right]\left[\frac{(-1)^{n-k}(-\mu)_{n-k}}{(n-k)!}\right] \\ &= \sum_{k=0}^{n} \left[\frac{(-1)^k(-\lambda)_k}{k!}\right]\left[\frac{(-1)^{n-k}(-1)^k(-\mu)_n}{(1+\mu-n)_k} \cdot \frac{(-n)_k}{(-1)^k n!}\right] \\ &= \sum_{k=0}^{n} \frac{(-1)^k(-\lambda)_k}{k!} \cdot \frac{(-1)^{n-k}(-\mu)_n(-n)_k}{(1+\mu-n)_k n!} \\ &= \frac{(-1)^n(-\mu)_n}{n!}\sum_{k=0}^{n} \frac{(-n)_k(-\lambda)_k}{(1+\mu-n)_k k!} \\ &= \frac{(-1)^n(-\mu)_n}{n!}\,{}_2F_1\,(-n\,, -\lambda;\, 1+\mu-n\,;\, 1), \end{aligned}$$

which, for $b = -\lambda$ and $c = 1 + \mu - n$, in view of (2.6), yields

$$S_n = \frac{(-1)^n\,(-\lambda-\mu)_n}{n!}.$$

This, in virtue of (1.10), completes the proof of (2.7).

Recall a transformation formula for ${}_2F_1$:

$$(2.8)\qquad {}_2F_1\,(c-a,\, c-b\,;\, c;\, z) = (1-z)^{a+b-c}\,{}_2F_1(a,\, b\,;\, c\,;\, z),$$

which is due to Euler. Rainville [4, pp. 86-88] obtained (2.3) by interpreting (2.8) as an identity involving three power series. Conversely, as suggested in Bailey [1, p. 245], we can prove (2.8) by making use of (2.3). Indeed, let

$$(2.9)\qquad (1-z)^{a+b-c}\,{}_2F_1(a,\,b\,;\,c\,;\,z) := \sum_{n=0}^{\infty} a_n z^n.$$

Starting with the left side of (2.9), we have

$$\begin{aligned}(1-z)^{a+b-c}\,{}_2F_1(a,\ b\,;\,c\,;\,z)\\ &= \left[\sum_{n=0}^{\infty} \frac{(c-a-b)_n}{n!} z^n\right]\left[\sum_{n=0}^{\infty} \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}\right]\\ &= \sum_{n=0}^{\infty}\left[\sum_{k=0}^{n} \frac{(c-a-b)_{n-k}(a)_k(b)_k}{(n-k)!(c)_k k!}\right] z^n,\end{aligned}$$

which, in view of (1.7) and (1.8) with equating the coefficients of z^n, yields

$$a_n = \frac{(c-a-b)_n}{n!}\,{}_3F_2\left[\begin{matrix} -n, & a, & b; \\ c, 1-c+a+b-n; \end{matrix}\ 1\right],$$

which, using (2.3), becomes

$$a_n = \frac{(c-a)_n(c-b)_n}{n!(c)_n}.$$

This completes the proof of (2.8).

Choi [2, p. 160] proved the following formula by mathematical induction method:

$$(2.10)\qquad \frac{(A)_n(B)_n}{(C)_n} = \sum_{k=0}^{n}\binom{n}{k}\frac{(C-B)_k(C-A)_k}{(C)_k}(A+B-C)_{n-k},$$

which can incidentally be seen to be equivalent to Saalschütz's theorem (2.3). Indeed, applying (1.7) and (1 9) to (2.10), we obtain

$$(2.11)\qquad {}_3F_2\left[\begin{matrix} -n, & C-A, & C-B\,; \\ C, 1+C-A-B-n\,; \end{matrix}\ 1\right] = \frac{(A)_n\,(B)_n}{(A+B-C)_n\,(C)_n},$$

which, for $C = c$, $C - A = a$ and $C - B = b$, immediately yields (2.3) and vice versa.

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