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A NOTE ON CONTIGUOUS FUNCTION RELATIONS

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1. Introduction and Preliminaries

In the theory of special functions, it is presumably safe to say that the hypergeometric series is one of the important functions. The so-called hypergeometric series is defined by

$$(1.1) \qquad {}_2F_1\,(a,b;c;z) = {}_2F_1\left[\begin{matrix} a, & b, & \\ & c; & z\end{matrix}\right] := \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!},$$

where a, b and c are arbitrary complex constants and $(\alpha)_n$ denotes the Pochhammer symbol (or the generalized factorial, since $(1)_n = n!$) defined by

$$(1.2) \quad (\alpha)_n = \begin{cases} \alpha(\alpha+1)\cdots(\alpha+n-1) & \text{if } n \in \mathbf{N} := \{1,\ 2,\ 3,\ \cdots\} \\ 1 & \text{if } n = 0. \end{cases}$$

From the fundamental functional relation of the Gamma function Γ, $\Gamma(z+1) = z\Gamma(z)$, we have

$$(1.3) \qquad (\alpha)_n = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)},$$

where Γ is the well-known Gamma function whose Weierstrass canonical product form is given by

$$(1.4) \qquad \{\Gamma(z)\}^{-1} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}},$$

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γ being the Euler-Mascheroni's constant defined by

$$\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \log n\right) \cong 0.577\,215\,664\cdots. \tag{1.5}$$

From definitions (1.2) and (1.3), we can easily deduce the following formulas:

$$\begin{aligned} (\alpha)_{n-k} &= \frac{(-1)^k(\alpha)_n}{(1-\alpha-n)_k}; \\ (-n)_k &= \begin{cases} \dfrac{(-1)^k n!}{(n-k)!} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n. \end{cases} \end{aligned} \tag{1.6}$$

In this paper we first give more contiguous relations besides 15 Gauss's contiguous relations among hypergeometric functions. We then apply some of these identities to obtain various identities involved in Jacobi polynomials. We also point out how some of those identities presented here were applied elsewhere.

2. The Contiguous Function Relations in ${}_2F_1$

Gauss defined as *contiguous* to $F(a, b, c; z)$ each of the six functions defined by increasing or decreasing one of the parameters by unity. For simplicity in printing, we use the notations

$$\begin{aligned} F &= {}_2F_1(a, b; c; z), \\ F(a+) &= F(a+1, b; c; z), \\ F(a-) &= F(a-1, b; c; z) \end{aligned} \tag{2.1}$$

together with similar notations $F(b+)$, $F(b-)$, $F(c+)$, and $F(c-)$.

Put $\delta_n = \frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!}$ so that $F = \sum_{n=0}^{\infty}\delta_n$ and

$$F(a+) = \sum_{n=0}^{\infty}\frac{(a+1)_n}{(a)_n}\delta_n = \sum_{n=0}^{\infty}\frac{a+n}{a}\delta_n. \tag{2.2}$$

By using differential operator $\theta = z\left(\frac{d}{dz}\right)$, we readily find that

$$(2.3)\qquad \begin{aligned} (\theta + a)F &= \sum_{n=0}^{\infty}(a+n)\delta_n, \\ \theta F &= z\sum_{n=0}^{\infty}\frac{(a+n)(b+n)}{(c+n)}\delta_n, \\ \theta F(a-) &= (a-1)z\sum_{n=0}^{\infty}\frac{b+n}{c+n}\delta_n. \end{aligned}$$

Gauss proved that between F and any two of its contiguous functions, there exists a linear relation with coefficients at most linear in z. He obtained 15 relations among contiguous hypergeometric functions (see Rainville [6, p. 71]).

Indeed, in view of (2.2) and (2.3), the following five relations are obtained:

$$(2.4)\qquad \begin{aligned} (a-b)F &= aF(a+) - bF(b+), \\ (a-c+1)F &= aF(a+) - (c-1)F(c-), \\ [a+(b-c)z]F &= a(1-z)F(a+) - c^{-1}(c-a)(c-b)zF(c+), \\ (1-z)F &= F(a-) - c^{-1}(c-b)zF(c+), \\ (1-z)F &= F(b-) - c^{-1}(c-a)zF(c+). \end{aligned}$$

We then obtain the remaining ten such relations by combining relations in (2.4):

$$\begin{aligned} [2a-c+(b-a)z]F &= a(1-z)F(a+) - (c-a)F(a-), \\ (a+b-c)F &= a(1-z)F(a+) - (c-b)F(b-), \\ (c-a-b)F &= (c-a)F(a-) - b(1-z)F(b+), \\ (b-a)(1-z)F &= (c-a)F(a-) - (c-b)F(b-), \\ [1-a+(c-b-1)z]F &= (c-a)F(a-) - (c-1)(1-z)F(c-), \\ [2b-c+(a-b)z]F &= b(1-z)F(b+) - (c-b)F(b-), \\ [b+(a-c)z]F &= b(1-z)F(b+) - c^{-1}(c-a)(c-b)z \\ &\quad \times F(c+), \end{aligned}$$

$$
(2.5)\quad
\begin{aligned}
&(b-c+1)F = bF(b+) - (c-1)F(c-),\\
&[1-b+(c-a-1)z]F = (c-b)F(b-) - (c-1)(1-z)F(c-),\\
&[c-1+(a+b+1-2c)z]F = (c-1)(1-z)F(c-)\\
&\quad - c^{-1}(c-a)(c-b)zF(c+).
\end{aligned}
$$

The notation used in (2.4) and (2.5) is extended as in the following examples

$$
\begin{aligned}
F(a-,\ b+) &:= F(a-1,\ b+1; c; z),\\
F(b+,\ c+) &:= F(a,\ b+1; c+1; z),
\end{aligned}
$$

and so on. Next, we obtain more contiguous function relations involving those just defined by making use of identities in (2.4) and (2.5):

$$
(2.6)\quad
\begin{aligned}
&F = F(a-,b+) + c^{-1}(b+1-a)(c-b)zF(b+,c+),\\
&(a-1)F = (a-b-1)F(a-) + bF(a-,b+),\\
&(b-1)F = (b-a+1)F(b-) + aF(a+,b-),\\
&(a-1)F = (a-c)F(a-) + (c-1)F(a-,c-),\\
&cF = (c-a)F(c+) + aF(a+,c+),\\
&(a-1)(1-z)F = [a-1+(b-c)z]F(a-) + c^{-1}(c-a+1)\\
&\quad \times (c-b)zF(a-,c+),\\
&\frac{(c-a-1)(c-b-1)zF}{(c-1)} = [(c-b-1)z-a]F(c-) + a(1-z)\\
&\qquad\qquad \times F(a+,c-),\\
&F = (1-z)F(a+) + c^{-1}(c-b)zF(a+,c+),\\
&(c-1)^{-1}(c-b-1)zF = (z-1)F(c-) + F(a-,c-),\\
&F = (1-z)F(b+) + c^{-1}(c-a)zF(b+,c+),\\
&(c-1)^{-1}(c-a-1)zF = (z-1)F(c-) + F(b-,c-),\\
&(a-1)(1-z)F = (a+b-c-1)F(a-) + (c-b)F(a-,b-),\\
&(c-b-1)F = (c-a-b-1)F(b+) + a(1-z)F(a+,b+),\\
&(c-a-1)F = (c-a-b-1)F(a+) + b(1-z)F(a+,b+),\\
&(b-1)(1-z)F = (a+b-c-1)F(b-) + (c-a)F(a-,b-),\\
&(c-a-1)F = (b-a-1)(1-z)F(a+) + (c-b)F(a+,b-),\\
&(c-b-1)F = (a-b-1)(1-z)F(b+) + (c-a)F(a-,b+),
\end{aligned}
$$

$$(c-a-1)F = [-a+(c-b-1)z]F(a+) + (c-1)(1-z)F(a+,c-),$$
$$c(1-z)F = [a-1-(c-b)z]F(c+) + (c-a+1)F(a-,c+),$$
$$(b-1)(1-z)F = [b-1+(a-c)z]F(b-) + c^{-1}(c-a)(c-b+1)z \times F(b-,c+),$$
$$\frac{(c-a-1)(c-b-1)zF}{(c-1)} = [-b-(a-c+1)z]F(c-) + b(1-z) \times F(b+,c-),$$
$$(b-1)F = (b-c)F(b-) + (c-1)F(b-,c-),$$
$$cF = (c-b)F(c+) + bF(b+,c+),$$
$$(c-b-1)F = [-b+(c-a-1)z]F(b+) + (c-1)(1-z)F(b+,c-),$$
$$c(1-z)F = [b-1-(c-a)z]F(c+) + (c-b+1)F(b-,c+).$$

The first identity of which was posed as a problem in [6, p. 72, Exercise 22]. Indeed, by using (2.3), we find two contiguous function relations

$$\theta F(a-) = (a-1)zF - c^{-1}(a-1)(c-b)zF(c+), \quad \theta F(a-) = (a-1)F - (a-1)F(a-). \tag{2.3.1}$$

In view of (2.3.1), we can easily find that

$$(1-z)F = F(a-) - c^{-1}(c-b)zF(c+). \tag{2.3.2}$$

Similarly, we may write

$$(1-z)F = F(b-) - c^{-1}(c-a)zF(c+). \tag{2.3.3}$$

We now have a contiguous function relation, together with (2.3.2) and (2.3.3),

$$(a-b)F = aF(a+) - bF(b+), \tag{2.4.1}$$

which, with replaced a by $a-1$, yields

$$(a-1-b)\sum_{n=0}^{\infty}\frac{(a-1)_n(b)_n}{(c)_n}\frac{z^n}{n!} = (a-1)\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\frac{z^n}{n!} - b\sum_{n=0}^{\infty}\frac{(a-1)_n(b+1)_n}{(c)_n}\frac{z^n}{n!},$$

from which we obtain

$$(a-1)F = (a-b-1)F(a-) + bF(a-, b+), \tag{2.6.1}$$

which is the second identity of (2.6).

Others can readily be obtained by using (2.3) and combining suitable identities in (2.4) and (2.5).

3. Applications

Consider Jacobi polynomials (see Szegö [9, pp. 23-29]) $P_n^{(\alpha,\beta)}(x)$ which may be defined by

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\ {}_2F_1\left(-n, 1+\alpha+\beta+n; 1+\alpha; \frac{1-x}{2}\right). \tag{3.1}$$

It follows, with the usual differential operator $D = d/dx$, that

$$DP_n^{(\alpha,\beta)}(x) = \frac{1}{2}(1+\alpha+\beta+n)P_{n-1}^{(\alpha+1,\beta+1)}(x). \tag{3.2}$$

By simple transformation, we found that

(3.3)

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x+1}{2}\right)^n\ {}_2F_1\left(-n, -\beta-n; 1+\alpha; \frac{x-1}{x+1}\right),$$

$$P_n^{(\alpha,\beta)}(x) = \frac{(1+\alpha)_n}{n!}\left(\frac{x-1}{2}\right)^n\ {}_2F_1\left(-n, -\alpha-n; 1+\beta; \frac{x+1}{x-1}\right).$$

If we use (3.2) and (3.3), we obtain

$$\begin{aligned}(\alpha+\beta+n+1)&P_n^{(\alpha,\beta+1)}(x) \\ &= (\beta+n+1)P_n^{(\alpha,\beta)}(x) + (\alpha+n)P_n^{(\alpha-1,\beta+1)}(x).\end{aligned} \tag{3.4}$$

Now we show how the principles presented in Section 2 can be applied to obtain some identities involving Jacobi polynomials. Consider the second identity of (2.6), that is, (2.6.2) :

$$\begin{aligned}(a-1)&F(a,b;c;z) \\ &= (a-b-1)F(a-1,b;c,z) + bF(a-1,b+1;c;z).\end{aligned}$$

Put $a = -n + 1$, $b = 1 + \alpha + \beta + n$, $c = 1 + \alpha$, $z = \frac{1}{2}(1 - x)$ in (2.6.2) to obtain

$$\begin{aligned}(3.5)\qquad &(\alpha+\beta+2n+1)P_n^{(\alpha,\beta)}(x)\\ &\quad = (\alpha+\beta+n+1)P_n^{(\alpha,\beta+1)}(x) + (\alpha+n)P_{n-1}^{(\alpha,\beta+1)}(x).\end{aligned}$$

Finally, combining (3.4) and (3.5) yields

$$(3.6)\qquad P_n^{(\alpha,\beta)}(x) = P_{n-1}^{(\alpha,\beta+1)}(x) + P_n^{(\alpha-1,\beta+1)}(x).$$

Similarly, using the last two contiguous function relations in (2.4), we obtain from (2.6.1),

$$F(a,b;c;z) = F(a-1,b+1;c;z) + c^{-1}(b+1-a)zF(a,b+1;c+1;z).$$

In view of (2.6.1), as in (3.6) (see also Rainville [6, pp. 263-265]), we obtain

$$(3.7)\qquad (1+x)P_n^{(\alpha,\beta+1)}(x) + (1-x)P_n^{(\alpha+1,\beta)}(x) = 2P_n^{(\alpha,\beta)}(x).$$

Choi *et al.* [5] applied the fourth identity of (2.4) and the first one of (2.6), respectively, to obtain two useful contiguous analogues of the Kummer's summation formula

$$(3.8)\qquad {}_2F_1\left[\begin{matrix} a, \quad b \quad ; \\ \frac{1}{2}(a+b+1); \end{matrix}\ \frac{1}{2}\right] = \frac{\Gamma(\frac{1}{2})\Gamma(\frac{1}{2}+\frac{1}{2}a+\frac{1}{2}b)}{\Gamma(\frac{1}{2}+\frac{1}{2}a)\Gamma(\frac{1}{2}+\frac{1}{2}b)}:$$

$$\begin{aligned}(3.9)\qquad {}_2F_1\left[\begin{matrix} a, \quad b \quad ; \\ \frac{1}{2}(a+b); \end{matrix}\ \frac{1}{2}\right] &= \Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\\ &\times\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})} + \frac{1}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}\right\}:\end{aligned}$$

$$\begin{aligned}(3.10)\qquad {}_2F_1\left[\begin{matrix} a, \quad b \quad ; \\ 1+\frac{1}{2}(a+b); \end{matrix}\ \frac{1}{2}\right] &= \frac{a+b}{a-b}\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}b\right)\\ &\times\left\{\frac{1}{\Gamma(\frac{1}{2}a)\Gamma(\frac{1}{2}b+\frac{1}{2})} - \frac{1}{\Gamma(\frac{1}{2}b)\Gamma(\frac{1}{2}a+\frac{1}{2})}\right\}.\end{aligned}$$

Vowe *et al.* [10] proved the following identity

$$\sum_{k=0}^{\infty}(-1)^k\binom{n-1}{k}\frac{1}{2^k(n+k+1)}=\frac{2^n(n-1)!\,n!}{(2n)!}-\frac{2^{-n}}{n}(n\in\mathbf{N}) \tag{3.11}$$

by evaluating the integral

$$\int_0^1\left(1-\frac{t}{2}\right)^{n-1}t^n dt. \tag{3.12}$$

We show the identity (3.11) by making use of one of the contiguous function relations presented here, and another form of Kummer's summation formula:

$${}_2F_1\left(a,1-a;b;\frac{1}{2}\right)=\frac{\Gamma(\frac{b}{2})\Gamma(\frac{b+1}{2})}{\Gamma(\frac{b+a}{2})\Gamma(\frac{b-a+1}{2})} \quad (b\neq 0,-1,-2,\cdots), \tag{3.13}$$

and Legendre duplication formula for the Gamma function

$$\Gamma\left(\frac{1}{2}\right)\Gamma(2z)=2^{2z-1}\Gamma(z)\Gamma\left(z+\frac{1}{2}\right). \tag{3.14}$$

Indeed, it is a routine work to see that the right side of (3.11) (say, S_n) is

$$S_n:=\frac{1}{n+1}\,{}_2F_1\left(-n+1,n+1\,;n+2\,;\frac{1}{2}\right). \tag{3.15}$$

Now, in view of the second formula in (2.4), we obtain

$$n\,{}_2F_1\left(1+n,1-n\,;n+2\,;\frac{1}{2}\right)=(1+n)\times\,{}_2F_1\left(n,1-n\,;1+n\,;\frac{1}{2}\right)-\,{}_2F_1\left(n,1-n\,;2+n\,;\frac{1}{2}\right). \tag{3.16}$$

Now first applying (3.13) to the first and second series in (3.16), and then using (3.14) in the resulting equation immediately leads to

$$S_n = \frac{2^n(n-1)!\, n!}{(2n)!} - \frac{2^{-n}}{n} \quad (n \in \mathbf{N}), \tag{3.17}$$

which is just our desired evaluation (3.11) (see also Srivastava [8]).

We conclude this note by noting the principles of Section 2 are no bound and so are their applications.

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References

[1] R Askey, *Theory and Application of Special Functions*, Proceedings of an Advanced Seminar Sponsored by the Mathematical Research Center (The University of Wisconsin Madison March 31- April 2, 1975), Academic Press, New York, San Francisco, London, 1975

[2] R Askey and G. Gasper, *Positive Jacobi polynomial sums*, Amer. Math J. **3-4** (1976), 709-737.

[3] H. Bateman, *Higher Transcendental Functions*, Vol I, McGraw-Hill Company, 1953

[4] J. Chou, *Note on Vandermonde's convolution theorem*, East Asian Math J. **14** (1998), 157-163

[5] J Choi, H M Srivastava, and Yongsup Kim, *Applications of a certain family of hypergeometric summation formulas associated with Psi and Zeta functions*, Preprint

[6] E. D Rainville, *Special Functions*, The Macmillan Company, New York, 1960

[7] H M Srivastava, *Sums of a certain family of series,*, Elem. Math **43** (1988), 54-58

[8] H. M Srivastava and P W Karlsson, *Multiple Gaussian Hypergeometric Series*, Ellis Horwood Limited (1985).

[9] G. Szegö, *Orthogonal Polynomials*, American Mathematical Society Colloquium Publications 23 (4th Ed) (1975).

[10] M Vowe, H. -J Seiffert, *Aufgabe 946*, Elem Math **42** (1987), 111-112

[11] E. T Whittaker and G N. Watson, *A Course of Modern Analysis*, Cambridge University Press (4th Ed) (1963)

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