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GENERALIZED SOBOLEV SPACES AND SOME RELATED PROBLEMS

YOUNG SIK PARK

1. Introduction

The generalized Sobolev space H_ω^s was defined and studied by Pahk and Kang [6] using ultradistribution theory of Beurling [1] and Bjorck [2]. The space H_ω^s, with a weight function ω possessing some suitable properties, is a generalization of the Sobolev space H^s.

Pathak [9] studied general Sobolev spaces $H_\omega^{s,p}$, $1 \le p \le \infty$, as a generalization of the space H_ω^s. In this case, $H_\omega^{s,2} = H_\omega^s$.

Roumieu [10] has also given an ultradistribution theory in which growth of derivatives of test functions are restricted by means of certain sequences.

A unification of the two theories can be found in Komatsu [4] and he derived a lot of results. The Beurling type spaces have been defined by Björck [2] in terms of a weight funtion $\omega : R^n \longrightarrow [0, \infty)$ under some assumptions.

Park [8] studied the generalized Sobolev spaces $W_{L^p}(\Omega; (M_k))$, $W_{L^p}(\Omega; [M_k])$ and relation between $D_{L^p}(\Omega; [M_k])$ and $D(\Omega; [M_k])$.

In this article we investigate some problems on the space $E_M(R^n)$ of ultradifferentiable functions of class M and that of $E_M(K)$of Whitney jets of class M on a compact set K in R^n. Also we consider the problems on $M = (M_k)_0^\infty$ especially when it satisfies (M.2) and $(M.3)'$.

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2. Some previous results and ultradifferentiable functions

Let $M = (M_k)_0^\infty$ be a sequence of positive numbers which satisfies some of the following conditions with $M_0 = 1$;

(M.1) $M_k^2 \le M_{k-1}M_{k+1}, k \in N$;

(M.2) There are constants $K > 0$ and $H > 1$ such that

$$M_k \le KH^k \min_{0\le l\le k} M_l M_{k-l}, k \in N_0 = N \cup \{0\};$$

(M.3) There is a constant $K > 0$ such that

$$\sum_{l=k+1}^{\infty} \frac{M_{l-1}}{M_l} \le Kk\frac{M_k}{M_{k+1}}, \quad k \in N;$$

(M.3)′
$$\sum_{k=1}^{\infty} \frac{M_{k-1}}{M_k} < \infty.$$

We write $m_k = \dfrac{M_k}{M_{k-1}}, k \in N$, and define $m(t) =$ the number of $m_k \le t, M(t) = \sup_k \log \dfrac{t^k}{M_k}$.

PROPOSITION 2.1. *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1). Then,*

(1) $M(t) = \int_0^t \frac{m(\lambda)}{\lambda} d\lambda$ *i.e.,* $\frac{dM}{dt} = \frac{m(t)}{t}$,

(2) $m(t) + M(t) \le M(et)$,

(3) *(M.1)* $\Leftrightarrow \{m_k\}$ *is an increasing sequence,*

(4) $M_k \le m_k^k$ *and* $M_j M_{k-j} \le M_k$ *for* $j \le k$.

(5) $M(s+t) \le M(2s) + M(2t), \quad s,t > 0$.

Proof. They are obvious, for details see Park [7].

We will assume, in addition to (M.1) and $(M.3)'$, that M satisfies the following conditions, where A is some positive constant.

(M.4) $\quad M_k \le A^k M_j M_{k-j}, \quad 0 \le j \le k.$

(M.5) $\quad M_{k+1}^k \le A^k M_k^{k+1}, \quad k \in N_0$

(M.6) $\quad kM_k^2 \le (k-1)M_{k-1}M_{k+1}, \quad k \ge 2.$

Note that (M.6) $\Rightarrow$ (M.1) and (M.2) $\Leftrightarrow$ (M.4). It is known by Bruna [3] that the condition (M.4) implies (and is in fact equivalent to the statement) that for each $q \in N$ there exist A_q and B_q such that
(1) $qM(t) \le M(A_q t) +$ constant, $t > 0$; (2) $M_{qk} \le B_q M_k^q, k \in N_0$.

THEOREM 2.2. *If, for each $q \in N$, there exists B_q such that $M_{qk} \leq B_q M_k^q$, then there exists A_q such that $qM(t) \leq M(A_q t), \quad t > 0$.*

Proof. Let $A_q = \sup_k B_q^{\frac{1}{qk}}$, then

$$qM(t) = q \sup_k \log \frac{t^k}{M_k} \leq \sup_k \log \frac{B_q t^{qk}}{M_{qk}} \leq \sup_k \log \frac{(A_q t)^k}{M_k} = M(A_q t).$$

The condition (M.5) and (4) in Proposition 2.1 imply that $m_{k+1} \leq AM_k^{\frac{1}{k}} \leq Am_k$ and m_k and $M_k^{\frac{1}{k}}$ are the same order. It also implies that $m(t)$ and $M(t)$ are of the same order in the sense that, together with $m(t) \leq M(et)$, we also have $M(t) \leq Am(B't) \leq AM(Bt)$ for some constants $A, B' > 0$, where $B = eB'$.

PROPOSITION 2.3.

(1) *(M.6) $\Leftrightarrow \frac{M_{k+1}}{kM_k}$ is increasing.*

(2) *(M.6) implies $(\frac{M_k}{k!})^2 \leq \frac{M_{k+1}}{(k+1)!}\frac{M_{k-1}}{(k-1)!}$.*
i.e., $N_k = \frac{M_k}{k!}$ is logarithmically convex. The converse is not true in general.

(3) *(M.4) implies $N_k \leq A^k N_j N_{k-j}$.*
i.e., N_k satisfies condition (M.4). The converse is not true in general.

(4) *(M.5) implies $N_{k+1}^k \leq A^k N_k^{k+1}$.*
i.e., N_k satisfies condition (M.5). The converse is not true in general.

Proof. They are obvious.

Suppose $M = (M_k)_0^\infty$ satisfies (M.1) and (M.3)$'$. Let $E_M(R^n)$ be the space of functions $f \in C^\infty(R^n)$ such that, for every compact set K in R^n,

$$P_{K,h}(f) = \sup_{\substack{\alpha \in N_0^n \\ x \in K}} \frac{|D^\alpha f(x)|}{h^{|\alpha|} M_{|\alpha|}}, D^\alpha = (\frac{\partial}{\partial x_1})^{\alpha_1} \cdots (\frac{\partial}{\partial x_n})^{\alpha_n},$$

is finite for some $h > 0$. The condition (M.3)$'$ guarantees that $E_M(R^n)$ is a non quasi-analytic class(see Mandelbrojt [5]).

THEOREM 2.4. *The space $E_M(R^n)$ of ultradifferentiable functions of class M is a Silva space, that is, inductive limit of Fréchet spaces such that the canonical mappings are compact.*

Proof. We define for $j \in N$, $E_{M,j}(R^n) = \{f \in C^\infty(R^n) :$ for every compact set K in R^n, $P_{K,j}(f) < \infty\}$, where the topology in $E_{M,j}(R^n)$ is defined by, for an increasing sequence $\{K_i\}$ of compact sets such that $\cup K_i = R^n$, the system of seminorms $\{P_{K_i,j} \ : \ i \in N\}$.

Then $E_{M,j}(R^n)$ is a Fréchet space and the canonical mappings $E_{M,j}(R^n) \hookrightarrow E_{M,j+1}(R^n)$ are compact. Therefore,

$$E_M(R^n) = \text{ind } lim_{j\to\infty} E_{M,j}(R^n).$$

3. Non-quasi-analyticity

Suppose that $M = (M_k)_0^\infty$ satisfies (M.1). Integrating by parts, we have

$$\sum_{m_k \le t} \frac{1}{m_k} = \int_0^t \frac{dm(\lambda)}{\lambda} = \frac{m(t)}{t} + \int_0^t \frac{m(\lambda)}{\lambda^2} d\lambda. \tag{3 1}$$

Hence we can prove by (3.1) the Carleman's theorem: $(M.3)' \Leftrightarrow (1) \Leftrightarrow$ $(2) \Leftrightarrow (3) \Leftrightarrow (4)$; (1) $\sum_{k=1}^\infty \frac{1}{m_k} < \infty$, (2) $\int_0^\infty \frac{m(\lambda)}{\lambda^2} d\lambda < \infty$, (3) $\int_0^\infty \frac{M(t)}{t^2} dt < \infty$, (4) $\sum_{k=0}^\infty \frac{1}{M_k^{\frac{1}{k}}} < \infty$.

Also $(M.3)'$ implies

$$\lim_{k\to\infty} \frac{k}{m_k} = 0 \Leftrightarrow \lim_{t\to\infty} \frac{m(t)}{t} = 0 \Rightarrow \lim_{t\to\infty} \frac{M(t)}{t} = 0.$$

PROPOSITION 3.1. *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1). If $lim_{k\to\infty} \frac{k}{m_k} = 0$, then*

$$\int_0^\infty \frac{M(t)}{t^2} dt = \int_0^\infty \frac{m(t)}{t^2} dt, \tag{3.2}$$

$$\int_0^\infty \frac{dm(t)}{t} = \int_0^\infty \frac{m(t)}{t^2} dt. \tag{3.3}$$

Proof. We can show that easily.

THEOREM 3.2. *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1). Then M satisfies $(M.3)'$ if and only if there is a constant A such that*

$$\int_t^\infty \frac{m(\lambda)}{\lambda^2} d\lambda \leq A + \frac{m(t)}{t} \text{ for } t \geq m_1. \tag{3.4}$$

Proof. Suppose that M satisfies $(M.3)'$. Then $\frac{m(\lambda)}{\lambda} \to 0$ as $\lambda \to \infty$. Hence by setting $k = m(t) \geq 1$, we have

$$\int_t^\infty \frac{m(\lambda)}{\lambda^2} d\lambda = \frac{m(t)}{t} + \int_{t+0}^\infty \frac{dm(\lambda)}{\lambda} = \frac{m(t)}{t} + \sum_{q=k+1}^\infty \frac{1}{m_q} \leq \frac{m(t)}{t} + A.$$

Conversely suppose that (3.4) holds. Let $m_{k_0} < m_{k_0+1} = \cdots = m_k \leq m_k + 1$. We have again $\lim_{\lambda\to\infty} \frac{m(\lambda)}{\lambda} = 0$. Hence if $m_{k_0} < t < m_k$, then we have

$$\sum_{q=k}^\infty \frac{1}{m_q} \leq \sum_{q=k_0+1}^\infty \frac{1}{m_q} = \int_t^\infty \frac{dm(\lambda)}{\lambda} = \int_t^\infty \frac{m(\lambda)}{\lambda^2} d\lambda - \frac{m(t)}{t} \leq A.$$

THEOREM 3 3(KOMATSU [4] PROPOSITION 4.4). *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1) Then M satisfies (M.3) if and only if there is a constant A such that*

$$\int_t^\infty \frac{m(\lambda)}{\lambda^2} d\lambda \leq (A+1)\frac{m(t)}{t} \text{ for } t \geq m_1. \tag{3.5}$$

THEOREM 3.4. *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1). Then M satisfies (M.3) if and only if*

$$\int_t^\infty \frac{dm(\lambda)}{\lambda} \leq A\frac{m(t)}{t} \text{ for } t \geq m_1. \tag{3.6}$$

Proof. Since $\int_t^\infty \frac{m(\lambda)}{\lambda^2} d\lambda = \frac{m(t)}{t} + \int_t^\infty \frac{dm(\lambda)}{\lambda}$, (M.3) $\Leftrightarrow \int_t^\infty \frac{dm(\lambda)}{\lambda} \leq A\frac{m(t)}{t}$ by Theorem 3.3

PROPOSITION 3.5. *Suppose that* $M = (M_k)_0^\infty$ *satisfies (M.1) and* $(M.3)'$. *Then we have the following relations:*

$$\int_0^t \frac{m(\lambda)}{\lambda^2}d\lambda = \frac{M(t)}{t} + \int_0^t \frac{M(\lambda)}{\lambda^2}d\lambda, \tag{3.7}$$

$$\int_t^\infty \frac{M(\lambda)}{\lambda^2}d\lambda = \frac{M(t)}{t} + \int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda, \tag{3.8}$$

and hence by (1) or (2) we have

$$\int_0^\infty \frac{M(\lambda)}{\lambda^2}d\lambda = \int_0^\infty \frac{m(\lambda)}{\lambda}d\lambda. \tag{3.9}$$

By (1) and (2), we have

$$\int_0^t \frac{m(\lambda) - M(\lambda)}{\lambda^2}d\lambda = \int_t^\infty \frac{M(\lambda) - m(\lambda)}{\lambda^2}d\lambda. \tag{3.10}$$

Proof. By simple calculating we can show the relations.

Integrating both sides of (3.5), we obtain for $t \geq m_1$ $\int_{m_1}^t d\mu \int_\mu^\infty \frac{m(\lambda)}{\lambda^2}d\lambda =$
$t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda + \int_{m_1}^t \frac{m(\lambda)}{\lambda}d\lambda - m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda$
$\leq (A+1)\int_{m_1}^t \frac{m(\lambda)}{\lambda}d\lambda.$
Hence we have the following relation (3.11).

PROPOSITION 3.6. *(3.11)*$\Leftrightarrow$ *(3.12)*$\Rightarrow$ *(3.13)* $\Rightarrow$ *(3.14):*

$$t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \leq A\int_{m_1}^t \frac{m(\lambda)}{\lambda}d\lambda + m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda. \tag{3.11}$$

$$t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \leq A[M(t) - M(m_1)] + m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda. \tag{3.12}$$

$$t\int_t^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \leq AM(t) + m_1\int_{m_1}^\infty \frac{m(\lambda)}{\lambda^2}d\lambda \tag{3.13}$$

for all $t \geq 0$.

$$t\int_t^\infty \frac{M(\lambda)}{\lambda^2}d\lambda \leq (A+1)M(t) + m_1\int_0^\infty \frac{M(\lambda)}{\lambda^2}d\lambda \tag{3.14}$$

for all $t \geq 0$.

Proof. By (3.8) and (3.9), (3.13)$\Rightarrow$ (3.14). The others are obvious.

4. Whitney jets of class M on K

The letters α, β will mean multi-indexes in N_0^n. For $\alpha = (\alpha_1, \cdots, \alpha_n)$, we write $\alpha! = \alpha_1! \cdots \alpha_n!$ and $|\alpha| = \alpha_1 + \cdots + \alpha_n$. Also, $\alpha \leq \beta$ stands for $\alpha_i \leq \beta_i (i = 1, \cdots, n)$ and, for $x \in R^n, x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Let K be a compact set in R^n. A *jet* in K is a multisequence $F = (f_\alpha)$ of continuous functions f_α on K. For a jet F, for $x, y \in K, z \in R^n, m \in N_0$ and $|\alpha| \leq m$, we put

$$(4.1) \qquad (T_x^m F)(z) = \sum_{|\alpha| \leq m} \frac{f_\alpha(x)}{\alpha!}(z-x)^\alpha,$$

$$(4.2) \qquad (R_x^m F)_\alpha(y) = f_\alpha(y) - \sum_{|\alpha+\beta| \leq m} \frac{f_{\alpha+\beta}(x)}{\beta!}(y-x)^\beta.$$

A jet F is called a *Whitney jet* on K if it satisfies, for all $m \in N_0$ and $|\alpha| \leq m$,

$$(4.3) \qquad |(R_x^m F)_\alpha(y)| = o(|x-y|^{m-|\alpha|})$$

for $x, y \in K$, as $|x-y| \to 0$. We write $C^\infty(K)$ for the space of *Whitney jets* on K.

Let $C^m(K), m \in N_0$, be the space of all m times continuously differentiable functions on K in the sense of Whitney i.e., $C^m(K) = \{F = (f_\alpha; |\alpha| \leq m) \mid F$ is an array of continuous functions f_α on K such that for each $|\alpha| \leq m$

$$\frac{|(R_x^m F)_\alpha(y)|}{|x-y|^{m-|\alpha|}} \quad \text{tends to zero uniformly as} |x-y| \to 0 \quad \text{in } K\}.$$

Define the norm of $F = (f_\alpha) \in C^m(K)$ by

$$\|F\|_{C^m(K)} = \sup_{|\alpha| \leq m} \|f_\alpha\|_{C(K)}.$$

Then $(C^m(K), \|\cdot\|_{C^m(K)})$ is a Banach space. The Fréchet space $C^\infty(K)$ is defined by

$$C^\infty(K) = \text{proj} \lim_{m \to \infty} C^m(K).$$

DEFINITION 4.1. *A jet $F = (f_\alpha)$ on K is called a Whitney jet of class M if it satisfies the conditions;*

$$|f_\alpha(x)| \le A h^{|\alpha|} M_{|\alpha|}, \quad \alpha \in N_0^n, \quad x \in K, \tag{4.4}$$

(4.5)

$$|(R_x^m F)_\alpha(y)| \le B \frac{|x-y|^{m-|\alpha|+1}}{(m-|\alpha|+1)!} h^{m+1} M_{m+1}, x, y \in K, m \in N_0, |\alpha| \le m$$

for some constants $A, B > 0$ and some $h > 0$. We write $E_M(K)$ for the space of Whitney jets of class M on K.

Bruna [3] showed that Whitney's extension theorem for $E_M(R^n)$:

THEOREM 4.2. *Suppose $M = (M_k)_0^\infty$ satisfies (M.1),(M.4),(M.5), (M.6) and (M.3). Then, for any $F \in E_M(K)$ there exits $\tilde{f} \in E_M(R^n)$ such that $D^\alpha \tilde{f}(x) = f_\alpha(x)$ for all $\alpha \in N_0$ and $x \in K$.*

THEOREM 4.3. *For a jet $F = (f_\alpha)$ on K, we define*

$$\|F\|_{K,h} = \sup_{\substack{x \in K \\ \alpha \in N_0^n}} \frac{|f_\alpha(x)|}{h^{|\alpha|} M_{|\alpha|}} + \inf\{B \mid \text{ constant } B \text{ satisfies (4.5)}\},$$

$$E_{M,h}(K) = \{F = (f_\alpha) \in E_M(K) : \|F\|_{K,h} < \infty\}.$$

Then $E_M(K) = \operatorname{ind}\lim_{h\to\infty} E_{M,h}(K)$.

Proof. If $h < h'$, then $\|F\|_{K,h} \ge \|F\|_{K,h'}$ and hence the canonical mappings $E_{M,h}(K) \hookrightarrow E_{M,h'}(K)$ are compact and $E_M(K) = \operatorname{ind}\lim_{h\to\infty} E_{M,h}(K)$.

Suppose that $M = (M_k)_0^\infty$ satisfies (M.1),$(M.3)'$,(M.4),(M.5) and (M.6). We define $N_k = \frac{M_k}{k!}$, $N(t) = \sup_k \log \frac{t^k}{N_k}$ and $H(t) = \sup_k \frac{k!}{t^k M_k} = \exp N(t^{-1})$.

THEOREM 4.4. *Suppose that $M = (M_k)_0^\infty$ satisfies (M.1), (M.3) and (M.5). For each $n \in N$, there exists a sequence $\{a_k^n\}$ such that*

(1) $\sum_k \frac{a_k^n}{a_{k+1}^n} \le 2, \quad a_0^n = 1.$

(2) $a_k^n \le H(B\epsilon_n)\epsilon_n^k M_k, \quad k \in N_0,$

where B is a constant that does not depend on n, $\epsilon_n = \frac{AnM_n}{M_{n+1}}$, and $A \geq 1$ is a fixed constant in (M.5).

Proof. The construction is simply modified one in [3].

(1) We define

$$a_k^n = \begin{cases} \epsilon_n^k M_k & \text{for} \quad k > n \\ n^k & \text{for} \quad k \leq n. \end{cases}$$

Then $a_n^n = n^n \leq \epsilon_n^n M_n$ by (M.5). Hence, using (M.3),

$$\sum_{k \geq n} \frac{a_k^n}{a_{k+1}^n} \leq \sum_{k \geq n} \frac{\epsilon_n^k M_k}{\epsilon_n^{k+1} M_{k+1}} \leq \epsilon_n^{-1} An \frac{M_n}{M_{n+1}} = 1.$$

Since $\sum_{k<n} \frac{n^k}{n^{k+1}} = \sum_{k<n} \frac{1}{n} = 1$, we have (1).

(2) For $k > n$, (2) is obvious since $H(t) \geq 1$. Since $A \geq 1$. we have

$$\begin{aligned} \sup_{k \leq n} \frac{a_k^n}{\epsilon_n^k M_k} &= \sup_{k \leq n} \frac{n^k M_{k+1}^k}{A^k n^k M_n^k M_k} \leq \sup_{k \leq n} \frac{M_{k+1}^k}{M_n^k M_k} = \frac{M_{n+1}^n}{M_n^{n+1}} \\ &= \frac{A^n n^n}{\epsilon_n^n M_n} \leq \frac{(m_0 A)^n n!}{\epsilon_n^n M_n} \leq H(B\epsilon_n), \end{aligned}$$

where $m_0 = \min\{m \in N : m \leq \frac{n}{\sqrt[n]{n!}}\}$, $B = \frac{1}{m_0 A}$.

References

[1] A. Beurling, *Lectures 4 and 5, A.M.S. Summer Institute(Standford), Quasi-analyticity and general distributions*, 1961

[2] G. Bjorck, *Linear partial differential operators and generalized distributions*, Ark. Mat 6, 1965, p. 351-407.

[3] J. Bruna, *An extension theorem of Whitney type for non quasi-analytic classes of functions*, J. London Math. Soc. 22(2) (1980), 495-505.

[4] H. Komatsu, *Ultradistributions I. Structure theorems and a Characterization*, Fac. Sci. Univ. Tokyo Sect. IA 20 (1973), 25-105

[5] S. Mandelbrojt, *Series adherentes, regularisation des suites, applications*, Gauthier-Villars, Paris, 1952

[6] D. H. Pahk and B. H. Kang, *Sobolev spaces in the generalized distribution spaces of Beurling type*, Tsukuba J. Math. 15(2) (1991), 325-334

[7] Y. S. Park, *Some associated functions with Gevrey-type sequences and tempered ultradistributions*, East Asian Math. J. **13**, (1997).

[8] Y. S. Park, *Generalized Sobolev space of Roumieu type and some related problems*, Comm. Korean Math. Soc 14(1) (1999), 111-119.
[9] R. S. Pathak, *Generalized Sobolev spaces and pseudo-differential operators on spaces of utlradistributions, Katata/Kyoto*, World Scientific Co., 1995, p. 343-368.
[10] C. Roumieu, *Ultra-distributions definies sur R^n et sur certains classes de varietes differentiables*, J Analyse Math. 10 (1962-63), 153-192.

Department of Mathematics
Pusan National University
Pusan 609-735, Korea
E-mail : ysikpark@hyowon.pusan.ac.kr