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ANTI FUZZY CHARACTERISTIC IDEALS OF A BCK–ALGEBRA

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The concept of fuzzy sets was introduced by Zadeh [8]. Since then these ideas have been applied to other algebraic structures such as semigroups, groups, rings, etc.. Jun et al. [6] introduced the notion of fuzzy characteristic subalgebras/ideals of a BCK-algebra. They proved that a fuzzy ideal μ of a BCK-algebra is a fuzzy characteristic ideal if and only if each level ideal of μ is a characteristic ideal S. M Hong and Y B. Jun [3] introduced the concept of fuzzy characteristic Γ-ideals of a Γ-ring, and they showed that a fuzzy characteristic Γ-ideal is characterized in terms of its level Γ-ideals. Recently, on the other hand, they also [2] defined the notions of anti fuzzy ideals of a BCK-algebra. The present author [5], modifying S. M. Hong and Y B Jun's idea, introduced anti fuzzy prime ideals of a commutative BCK-algebra, and proved that every anti fuzzy prime ideal of a commutative BCK-algebra is an anti fuzzy ideal.

In this paper, we define the notion of anti fuzzy characteristic ideals of BCK-algebras, and obtain some results about it.

We begin with several preliminaries definitions and propositions.

DEFINITION 1. An algebra $(X, *, 0)$ of type (2,0) is called a *BCK-algebra* if it satisfies the following axioms : for all $x, y, z \in X$,

(a) $((x * y) * (x * z)) * (z * y) = 0$,
(b) $(x * (x * y)) * y = 0$,
(c) $x * x = 0$,
(d) $0 * x = 0$,
(e) $x * y = 0$ and $y * x = 0$ imply $x = y$.

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A BCK-algebra can be (partially) ordered by $x \leq y$ if and only if $x * y = 0$. This ordering is called *BCK-ordering.*

PROPOSITION 1. *In any BCK-algebra X, the following hold: for all $x, y, z \in X$,*

(1) $x * 0 = x$,
(2) $(x * y) * z = (x * z) * y$,
(3) $x * y \leq x$,
(4) $(x * y) * z \leq (x * z) * (y * z)$,
(5) $x \leq y$ *implies* $x * z \leq y * z$ *and* $z * y \leq z * x$.

DEFINITION 2. [4] A non-empty subset I of a BCK-algebra X is called an *ideal* of X if

(1) $0 \in I$,
(2) $x * y \in I$ and $y \in I$ imply $x \in I$.

DEFINITION 3. [8] Let S be a non-empty set A *fuzzy subset* μ of S is a function $\mu : S \to [0, 1]$.

DEFINITION 4. [1] Let μ be a fuzzy subset of S. Then for $t \in [0, 1]$, the *level subset* of μ is the set $\mu_t = \{x \in S \mid \mu(x) \geq t\}$.

DEFINITION 5. [7] Let X be a BCK-algebra. A fuzzy subset μ of X is called a *fuzzy subalgebra* of X if

$$\mu(x * y) \geq \min\{\mu(x), \mu(y)\}$$

for all $x, y \in X$.

DEFINITION 6. [7] Let X be a BCK-algebra. A fuzzy subset μ of X is called a *fuzzy ideal* of X if, for $x, y \in X$,

(1) $\mu(0) \geq \mu(x)$,
(2) $\mu(x) \geq \min\{\mu(x * y), \mu(y)\}$.

DEFINITION 7. [2] A fuzzy subset μ of a BCK-algebra X is called an *anti fuzzy subalgebra* of X if

$$\mu(x * y) \leq \max\{\mu(x), \mu(y)\}$$

for all x, $y \in X$.

PROPOSITION 2. [2] *Let μ be an anti fuzzy subalgebra of a BCK-algebra X. Then $\mu(0) \leq \mu(x)$ for every $x \in X$.*

DEFINITION 8. [2] A fuzzy subset μ of a BCK-algebra X is called an *anti fuzzy ideal* of X if

(1) $\mu(0) \leq \mu(x)$,
(2) $\mu(x) \leq \max\{\mu(x * y), \mu(y)\}$,

for all $x, y \in X$.

Clearly, every anti fuzzy ideal μ of a BCK-algebra X is an anti fuzzy subalgebra of X, but not conversely.

DEFINITION 9. If μ is a fuzzy subset of X and α is a function from X into itself, we define a function μ^α from X into $[0,1]$ by $\mu^\alpha(x) = \mu(\alpha(x))$ for every $x \in X$.

Suppose that μ is an anti fuzzy subalgebra of a BCK-algebra X and α is an endomorphism of X. Then

$$\begin{aligned}\mu^\alpha(x * y) &= \mu(\alpha(x * y)) \\ &= \mu(\alpha(x) * \alpha(y)) \\ &\leq \max\{\mu(\alpha(x)), \mu(\alpha(y))\} \\ &= \max\{\mu^\alpha(x), \mu^\alpha(y)\},\end{aligned}$$

for all $x, y \in X$ and

$$\begin{aligned}\mu^\alpha(x) &= \mu(\alpha(x)) \\ &= \max\{\mu(\alpha(x)), \mu(\alpha(x))\} \\ &\geq \mu(\alpha(x) * \alpha(x)) \\ &= \mu(\alpha(x * x)) \\ &= \mu^\alpha(0),\end{aligned}$$

for every $x \in X$. Hence we have the following proposition.

PROPOSITION 3. *Let μ be an anti fuzzy subalgebra of a BCK-algebra X and let α be an endomorphism of X. Then*

(1) *μ^α is an anti fuzzy subalgebra of X,*
(2) *$\mu^\alpha(0) \leq \mu^\alpha(x)$ for every $x \in X$.*

PROPOSITION 4. *Let μ be an anti fuzzy ideal of X and let α be an endomorphism of X. Then the following holds for all $x, y, z \in X$,*

(1) *if $x \leq y$, then $\mu^\alpha(x) \leq \mu^\alpha(y)$.*
(2) $\mu^\alpha(x * y) \leq \max\{\mu^\alpha(x * z), \mu^\alpha(z * y)\}$.
(3) *if $\mu^\alpha(x * y) = \mu^\alpha(0)$, then $\mu^\alpha(x) \leq \mu^\alpha(y)$.*
(4) $\max\{\mu^\alpha(x * y), \mu^\alpha(y)\} = \max\{\mu^\alpha(x), \mu^\alpha(y)\}$.
(5) *if X is bounded, then $\max\{\mu^\alpha(x), \mu^\alpha(1 * x)\} = \mu^\alpha(1)$.*
(6) *if $x \leq y$, then $\mu^\alpha(y) = \max\{\mu^\alpha(y * x), \mu^\alpha(x)\}$.*

Proof. (1) If $x \leq y$, then we have $x * y = 0$. Thus,

$$\begin{aligned}\mu^\alpha(x) &= \mu(\alpha(x)) \\ &\leq \max\{\mu(\alpha(x) * \alpha(y)), \mu(\alpha(y))\} \\ &= \max\{\mu(0), \mu(\alpha(y))\} \\ &= \mu(\alpha(y)) = \mu^\alpha(y).\end{aligned}$$

(2) From (a) of definition of BCK-algebra and (1), we have that $\mu^\alpha((x * y) * (x * z)) \leq \mu^\alpha(z * y)$. Thus,

$$\begin{aligned}\mu^\alpha(x * y) &= \mu(\alpha(x * y)) \\ &= \mu(\alpha(x) * \alpha(y)) \\ &\leq \max\{\mu((\alpha(x) * \alpha(y)) * (\alpha(x) * \alpha(z))), \mu(\alpha(x) * \alpha(z))\} \\ &= \max\{\mu^\alpha((x * y) * (x * z)), \mu^\alpha(x * z)\} \\ &\leq \max\{\mu^\alpha(z * y), \mu^\alpha(x * z)\}.\end{aligned}$$

(3) Suppose that $\mu^\alpha(x * y) = \mu^\alpha(0)$. Then

$$\begin{aligned}\mu^\alpha(x) &= \mu(\alpha(x)) \\ &\leq \max\{\mu(\alpha(x) * \alpha(y)), \mu(\alpha(y))\} \\ &= \max\{\mu(\alpha(x * y)), \mu(\alpha(y))\} \\ &= \max\{\mu^\alpha(x * y), \mu^\alpha(y)\} \\ &= \max\{\mu^\alpha(0), \mu^\alpha(y)\} \\ &= \max\{\mu(\alpha(0)), \mu(\alpha(y))\} \\ &= \max\{\mu(0), \mu(\alpha(y))\} \\ &= \mu(\alpha(y)) \\ &= \mu^\alpha(y).\end{aligned}$$

(4) Since $x * y \leq x$, we have $\mu^\alpha(x * y) \leq \mu^\alpha(x)$ by (1). On the other hand,

$$\begin{aligned}\mu^\alpha(x) &= \mu(\alpha(x)) \\ &\leq \max\{\mu(\alpha(x) * \alpha(y)), \mu(\alpha(y))\} \\ &= \max\{\mu(\alpha(x * y)), \mu(\alpha(y))\} \\ &= \max\{\mu^\alpha(x * y), \mu^\alpha(y)\}.\end{aligned}$$

It follows that $\max\{\mu^\alpha(x * y), \mu^\alpha(y)\} = \max\{\mu^\alpha(x), \mu^\alpha(y)\}$.

(5) If X is bounded, then by (1), $\mu^\alpha(1) \geq \max\{\mu^\alpha(x), \mu^\alpha(1 * x)\}$. On the other hand,

$$\begin{aligned}\mu^\alpha(1) &= \mu(\alpha(1)) \\ &\leq \max\{\mu(\alpha(1) * \alpha(x)), \mu(\alpha(x))\} \\ &= \max\{\mu(\alpha(1 * x)), \mu(\alpha(x))\} \\ &= \max\{\mu^\alpha(1 * x), \mu^\alpha(x)\}.\end{aligned}$$

Hence (5) holds.

(6) is obtained from (1) and (4).

PROPOSITION 5. *Let μ be an anti fuzzy ideal of X and let $\alpha : X \to X$ be an onto homomorphism. Then μ^α is an anti fuzzy ideal of X.*

Proof. For all $x \in X$, we have that

$$\mu^\alpha(x) = \mu(\alpha(x)) \geq \mu(0) = \mu(\alpha(0)) = \mu^\alpha(0).$$

Next for any $x, y \in X$,

$$\mu^\alpha(x) = \mu(\alpha(x)) \leq \max\{\mu(\alpha(x) * y), \mu(y)\}.$$

Since α is onto, there is $z \in X$ such that $\alpha(z) = y$. It follows that

$$\begin{aligned}\mu^\alpha(x) &\leq \max\{\mu(\alpha(x) * y), \mu(y)\} \\ &= \max\{\mu(\alpha(x) * \alpha(z)), \mu(\alpha(z))\} \\ &= \max\{\mu(\alpha(x * z)), \mu(\alpha(z))\} \\ &= \max\{\mu^\alpha(x * z), \mu^\alpha(z)\}.\end{aligned}$$

Since y is an arbitrary element of X, the above result is true for all $z \in X$, i.e., $\mu^\alpha(x) \leq \max\{\mu^\alpha(x * z), \mu^\alpha(z)\}$ for all $x, z \in X$. Thus μ^α is an anti fuzzy ideal of X.

Definition 10. An anti fuzzy subalgebra (ideal) μ of X is called an *anti fuzzy characteristic subalgebra (ideal)* of X if $\mu(\alpha(x)) = \mu(x)$ for all $x \in X$ and all $\alpha \in \mathrm{Aut}(X)$.

Definition 11. [2] Let μ be a fuzzy subset of a BCK-algebra X. Then for $t \in [0,1]$, the set

$$\mu^t := \{x \in X \mid \mu(x) \leq t\}$$

is called the *lower t-level cut* of μ.

Proposition 6. [2] *Let μ be a fuzzy subset of a BCK-algebra X. Then it is an anti fuzzy ideal of X if and only if for every $t \in [0,1]$, $t \geq \mu(0)$, the lower t-level cut μ^t is an ideal of X.*

Proposition 7. *Let μ be an anti fuzzy characteristic subalgebra of a BCK-algebra X. Then each lower t-level cut of μ is a characteristic subalgebra of X.*

Proof. Let $t \in \mathrm{Im}(\mu)$, $\alpha \in \mathrm{Aut}(X)$ and $x \in \mu^t$. Since μ is an anti fuzzy characteristic subalgebra of X, we have $\mu(\alpha(x)) = \mu(x) \leq t$. It follows that $\alpha(x) \in \mu^t$ and hence $\alpha(\mu^t) \subseteq \mu^t$. To show the reverse inclusion, let $x \in \mu^t$ and let $y \in X$ be such that $\alpha(y) = x$. Then $\mu(y) = \mu(\alpha(y)) = \mu(x) \leq t$, so $y \in \mu^t$. It follows that $x = \alpha(y) \in \alpha(\mu^t)$. Hence $\mu^t \subseteq \alpha(\mu^t)$. Thus μ^t is a characteristic subalgebra of X, for each $t \in \mathrm{Im}(\mu)$.

The proof of the following lemma is obvious, and we omit the proof.

Lemma 1. *Let μ be an anti fuzzy subalgebra(ideal) of X and let $x \in X$. Then $\mu(x) = t$ if and only if $x \in \mu^t$ and $x \notin \mu^s$ for all $s < t$.*

Now we consider the converse of Proposition 7.

Proposition 8. *Let μ be an anti fuzzy subalgebra of X. If each lower t-level cut μ^t is a characteristic subalgebra of X, then μ is an anti fuzzy characteristic subalgebra of X.*

Proof. Let $x \in X$, $\alpha \in \mathrm{Aut}(X)$ and $\mu(x) = t$. Then $x \in \mu^t$ and $x \notin \mu^s$ for all $s < t$, by Lemma 1. Since $\alpha(\mu^t) = \mu^t$ by hypothesis, we have $\alpha(x) \in \mu^t$. Hence $\mu(\alpha(x)) \leq t$. Let $s = \mu(\alpha(x))$. We now show that $s = t$. Indeed, suppose that $s < t$. Then $\alpha(x) \in \mu^s = \alpha(\mu^s)$.

Since α is one-to-one, we have $x \in \mu^s$. This is a contradiction. Thus $\mu(\alpha(x)) = t = \mu(x)$. It follows that μ is an anti fuzzy characteristic subalgebra of X.

The proofs of the following propositions are similar to those of Propositions 7 and 8.

PROPOSITION 9. *If μ is an anti fuzzy characteristic subalgebra of X, then each lower t-level cut of μ is a characteristic ideal of X.*

PROPOSITION 10. *Let μ be an anti fuzzy ideal of X. If each lower t-level cut of μ is a characteristic ideal of X, then μ is an anti fuzzy characteristic ideal of X.*

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