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AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES ARE BOUNDED AND APPLICATIONS

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1. Introduction

In 1938, Ostrowski (see for example [2, p. 468]) proved the following integral inequality:

THEOREM 1.1. *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \le \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty \tag{1.1}$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

For some applications of Ostrowski's inequality to some special means and numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for n-time differentiable mappings (see for example [2, p.468]) from which we would like to mention only the case of twice differentiable mappings [2, p. 470].

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THEOREM 1.2. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping such that $f'' : (a,b) \to \mathbb{R}$ is bounded on (a,b), i.e., $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\left|\frac{1}{2}\left[f(x) + \frac{(x-a)f(a)+(b-x)f(b)}{b-a}\right] - \frac{1}{b-a}\int_a^b f(t)dt\right| \leq \frac{\|f''\|_\infty}{4}(b-a)^2\left[\frac{1}{12} + \frac{\left(x-\frac{a+b}{2}\right)^2}{(b-a)^2}\right] \tag{1.2}$$

for all $x \in [a,b]$.

In this paper we point out an inequality of Ostrowski's type which is similar, in a sense, to Milovanović-Pečarić result and apply it for special means and in numerical integration.

2. Some Integral Inequalities

The following result holds.

THEOREM 2.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) and $f'' : (a,b) \to \mathbb{R}$ is bounded, i.e., $\|f''\|_\infty = \sup_{t\in(a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\begin{aligned}\left|f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x-\frac{a+b}{2}\right)f'(x)\right| &\leq \left[\frac{1}{24}(b-a)^2 + \frac{1}{2}\left(x-\frac{a+b}{2}\right)^2\right]\|f''\|_\infty \\ &\leq \frac{(b-a)^2}{6}\|f''\|_\infty\end{aligned} \tag{2.1}$$

for all $x \in [a,b]$.

Proof. Let us define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbb{R}$ given by

$$K(x,t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in (x,b]. \end{cases}$$

Integrating by parts, we have successively

$$
\begin{aligned}
\int_a^b K(x,t) f''(t)dt &= \int_a^x \frac{(t-a)^2}{2} f''(t)dt + \int_x^b \frac{(t-b)^2}{2} f''(t)dt \\
&= \frac{(t-a)^2}{2} f'(t)\Big|_a^x - \int_a^x (t-a)f'(t)dt + \frac{(t-b)^2}{2} f'(t)\Big|_x^b \\
&\quad - \int_x^b (t-b)f'(t)dt \\
&= \frac{(x-a)^2}{2} f'(x) - \left[(t-a)f(t)\Big|_a^x - \int_a^x f(t)dt\right] \\
&\quad - \frac{(b-x)^2}{2} f'(x) - \left[(t-b)f(t)\Big|_x^b - \int_x^b f(t)dt\right] \\
&= \frac{1}{2}\left[(x-a)^2 - (b-x)^2\right] f'(x) - (x-a)f(x) \\
&\quad + \int_a^x f(t)dt + (x-b)f(x) + \int_x^b f(t)dt \\
&= (b-a)(x - \frac{a+b}{2})f'(x) - (b-a)f(x) + \int_a^b f(t)dt
\end{aligned}
$$

from which we get the integral identity :

$$
\begin{aligned}
\int_a^b f(t)\,dt = (b-a)\,f(x) - (b-a)\left(x - \frac{a+b}{2}\right) f'(x) \\
+ \int_a^b K(x,t)\,f''(t)dt
\end{aligned}
\tag{2.2}
$$

for all $x \in [a,b]$.

Using the identity (2 2), we have

$$
\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\
&= \frac{1}{b-a}\left|\int_a^b K(x,t) f''(t)dt\right| \\
&\leq \frac{1}{b-a}\|f''\|_\infty \int_a^b |K(x,t)|dt
\end{aligned}
\tag{2.3}
$$

$$= \frac{1}{b-a}\|f''\|_\infty \left[\int_a^x \frac{(t-a)^2}{2}dt + \int_x^b \frac{(t-b)^2}{2}dt\right]$$
$$= \frac{1}{b-a}\|f''\|_\infty \left[\frac{(t-a)^3}{6}\Big|_a^x + \frac{(t-b)^3}{6}\Big|_x^b\right]$$
$$= \frac{1}{b-a}\|f''\|_\infty \left[\frac{(x-a)^3+(b-x)^3}{6}\right].$$

Now, observe that

$$(x-a)^3+(b-x)^3 = (b-a)\left[(x-a)^2+(b-x)^2-(x-a)(b-x)\right]$$
$$= (b-a)[(x-a+b-x)^2-3(x-a)(b-x)]$$
$$= (b-a)[(b-a)^2+3[x^2-(a+b)x+ab]]$$
$$= (b-a)\left[(b-a)^2+3\left[\left(x-\frac{a+b}{2}\right)^2-\left(\frac{b-a}{2}\right)^2\right]\right]$$
$$= (b-a)\left[\left(\frac{b-a}{2}\right)^2+3\left(x-\frac{a+b}{2}\right)^2\right].$$

Using (2.3), we get the desired inequality (2.1).

COROLLARY 2.2. *Under the above assumptions, we have the mid-point inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{(b-a)^2}{24}\|f''\|_\infty.$$

This follows by Theorem 2.1, choosing $x=\frac{a+b}{2}$.

COROLLARY 2.3. *Under the above assumptions we have the following trapezoid like inequality:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{4}(f'(b)-f'(a))\right|$$
$$\leq \frac{(b-a)^2}{6}\|f''\|_\infty.$$

This follows using Theorem 2.1 with $x=a$, $x=b$, adding the results and using the triangle inequality for the modulus.

3. Applications in Numerical Integration

Let $I_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ be a division of the interval $[a,b]$, $\xi_i \in [x_i, x_{i+1}]$ $(i = 0,...,n-1)$. We have the following quadrature formula:

THEOREM 3.1. *Let $f : [a,b] \to \mathbb{R}$ be a twice differentiable mapping on (a,b) whose second derivative $f'' : (a,b) \to \mathbb{R}$ is bounded, i.e., $\|f''\|_\infty < \infty$. Then we have the following :*

$$\int_a^b f(x)dx = A(f,f',\xi,I_n) + R(f,f',\xi,I_n) \tag{3.1}$$

where

$$A(f,f',\xi,I_n) = \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder satisfies the estimation .
(3.2)
$$\begin{aligned}|R(f,f',\xi,I_n)| &\le \left[\frac{1}{24}\sum_{i=0}^{n-1} h_i^3 + \frac{1}{2}\sum_{i=0}^{n-1} h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f''\|_\infty \\ &\le \frac{\|f''\|}{6}\sum_{i=0}^{n-1} h_i^3\end{aligned}$$

for all ξ_i as above, where $h_i := x_{i+1} - x_i$ $(i = 0,...,n-1)$.

Proof. Apply Theorem 2.1 on the interval $[x_i, x_{i+1}]$ $(i = 0,...,n-1)$ to get

$$\begin{aligned}&\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - h_i f(\xi_i) + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i f'(\xi_i)\right| \\ &\le \left[\frac{1}{24}h_i^3 + \frac{1}{2}h_i\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right)^2\right]\|f''\|_\infty \le \frac{\|f''\|_\infty}{6}h_i^3.\end{aligned}$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce the desired estimation.

REMARK 3.2. *Choosing* $\xi_i = \frac{x_i + x_{i+1}}{2}$, *we recapture the midpoint quadrature formula*

$$\int_a^b f(x)\,dx = A_M(f, I_n) + R_M(f, I_n)$$

where the remainder $R_M(f, I_n)$ *satisfies the estimation*

$$|R_M(f, I_n)| \leq \frac{\|f''\|_\infty}{24} \sum_{i=0}^{n-1} h_i^3.$$

4. Applications for Special Means

Let us recall the following means :

(a) The arithmetic mean

$$A = A(a,b) := \frac{a+b}{2}, \qquad a, b \geq 0;$$

(b) The geometric mean:

$$G = G(a,b) := \sqrt{ab}, \qquad a, b \geq 0;$$

(c) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, \qquad a, b \geq 0;$$

(d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if } a = b; \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b, \end{cases}$$

where $a, b > 0$.

(e) The identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if } a = b; \\ \frac{1}{e}\left(\frac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b, \end{cases}$$

where $a, b > 0$.

(f) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b, \end{cases}$$

where $p \in \mathbb{R} \backslash \{-1, 0\}, \quad a, b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbb{R}$ with $L_0 = I$ and $L_{-1} = L$.

(1). Consider the mapping $f : (0, \infty) \to \mathbb{R}$, $f(x) = x^r$, $r \in \mathbb{R} \backslash \{-1, 0\}$.

Then, we have, for $0 < a < b$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_r^r(a,b)$$

and

$$\|f''\|_\infty = |r(r-1)|\,\delta_r(a,b), \qquad r \in \mathbb{R} \backslash \{-1, 0\},$$

where

$$\delta_r(a,b) := \begin{cases} b^{r-1} & \text{if } r \in (1, \infty) \\ a^{r-1} & \text{if } r \in (-\infty, 1) \backslash \{-1, 0\}. \end{cases}$$

Using the inequality (2.1) we have the result:

$$\begin{aligned} &\left|x^r - L_r^r(a,b) - r(x-A)x^{r-1}\right| \\ &\leq \frac{|r(r-1)|}{6}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right]\delta_r(a,b) \\ &\leq \frac{|r(r-1)|(b-a)^2}{6}\delta_r(a,b) \end{aligned} \tag{4.1}$$

for all $x \in [a,b]$. If in (4.1) we choose $x = A$, we get

$$|A^r - L_r^r| \le \frac{|r(r-1)|(b-a)^2}{24}\delta_r(a,b). \tag{4.2}$$

(2). Consider the mapping $f(x) = \frac{1}{x}$, $\quad x \in [a,b] \subset (0,\infty)$. Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_{-1}^{-1}(a,b)$$

and

$$\|f''\|_\infty = \frac{2}{a^3}.$$

Applying the inequality (2.1) for the above mapping, we get

$$\begin{aligned}\left|\frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2}\right| &\le \frac{1}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right] \\ &\le \frac{(b-a)^2}{3a^3}\end{aligned}$$

which is equivalent to

$$\begin{aligned}|x(L-x) - L(A-x)| &\le \frac{x^2L}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right] \\ &\le \frac{x^2L(b-a)^2}{3a^3}\end{aligned} \tag{4.3}$$

for all $x \in [a,b]$. Now, if we choose in (4.3), $x = A$, we get

$$0 \le A - L \le \frac{(b-a)^2 AL}{12a^3}. \tag{4.4}$$

If in (4.3) we choose $x = L$, we get

$$0 \le A - L \le \frac{L^2}{3a^3}\left[\frac{1}{4}(b-a)^2 + 3(L-A)^2\right]. \tag{4.5}$$

(3). Let us consider the mapping

$$f(x) = \ln x, \qquad x \in [a,b] \subset (0,\infty).$$

Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \ln I(a,b),$$

and

$$\|f''\|_\infty = \frac{1}{a^2}.$$

Inequality (2.1) gives us

$$\left|\ln x - \ln I - \frac{x-A}{x}\right| \le \frac{1}{6a^2}\left[\frac{1}{4}(b-a)^2 + 3(x-A)^2\right] \le \frac{(b-a)^2}{6a^2}. \tag{4.6}$$

Now, if in (4.6) we choose $x = A$, we get

$$1 \le \frac{A}{I} \le \exp\left[\frac{1}{24a^2}(b-a)^2\right]. \tag{4.7}$$

If in (4.6) we choose $x = I$, we get

$$0 \le A - I \le \frac{I}{6a^2}\left[\frac{1}{4}(b-a)^2 + 3(A-I)^2\right]. \tag{4.8}$$

References

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