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AN INEQUALITY OF OSTROWSKI TYPE FOR MAPPINGS WHOSE SECOND DERIVATIVES BELONG TO $L_1(a,b)$ AND APPLICATIONS

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Abstract An inequality of Ostrowski type for twice differentiable mappings whose derivatives belong to $L_1(a,b)$ and applications in Numerical Integration and for special means (logarithmic mean, identric mean, p-logarithmic mean etc...) are given.

1. Introduction

In 1938, Ostrowski (see for example [3, p. 468]) proved the following integral inequality:

THEOREM 1. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping on I° (I° is the interior of I), and let $a, b \in I^\circ$ with $a < b$. If $f' : (a,b) \to \mathbf{R}$ is bounded on (a,b), i.e., $\|f'\|_\infty := \sup_{t\in(a,b)} |f'(t)| < \infty$, then we have the inequality:*

(1.1)
$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)dt \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a)\|f'\|_\infty$$

for all $x \in [a,b]$. The constant $\frac{1}{4}$ is sharp in the sense that it can not be replaced by a smaller one.

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For some applications of Ostrowski's inequality to some special means and some numerical quadrature rules, we refer to the recent paper [1] by S.S. Dragomir and S. Wang.

In paper [2], the same authors considered another inequality of Ostrowski type for $\|\cdot\|_1$ norm as follows:

THEOREM 2. *Let $f : I \subseteq \mathbf{R} \to \mathbf{R}$ be a differentiable mapping in I° and $a, b \in I^\circ$ with $a < b$. If $f' \in L_1[a, b]$, then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \leq \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{(b-a)} \right] \|f'\|_1 \tag{1.2}$$

for all $x \in [a, b]$.

They also pointed out some applications of (1.2) in Numerical Integration as well as for special means.

In 1976, G.V. Milovanović and J.E. Pečarić proved a generalization of Ostrowski's inequality for n-time differentiable mappings (see for example [3, p. 468]) from which we would like to mention only the case of twice differentiable mappings [3, p. 470]:

THEOREM 3. *Let $f : [a, b] \to \mathbf{R}$ be a twice differentiable mapping such that $f'' : (a, b) \to \mathbf{R}$ is bounded on (a, b), i.e., $\|f''\|_\infty = \sup_{t \in (a,b)} |f''(t)| < \infty$. Then we have the inequality:*

$$\begin{aligned} &\left| \frac{1}{2}\left[f(x) + \frac{(x-a)f(a) + (b-x)f(b)}{b-a} \right] - \frac{1}{b-a}\int_a^b f(t)\,dt \right| \\ &\leq \frac{\|f''\|_\infty}{4}(b-a)^2 \left[\frac{1}{12} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] \end{aligned} \tag{1.3}$$

for all $x \in (a, b)$.

In this paper we point out an inequality of Ostrowski type for twice differentiable mappings which is in terms of the $\|\cdot\|_1$ -norm of the second derivative f'' and apply it in numerical integration and for some special means such as : logarithmic mean, identric mean, p-logarithmic mean etc.

2. Some Integral Inequalities

The following inequality of Ostrowski's type for mappings which are twice differentiable, holds.

THEOREM 4. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$, twice differentiable on (a,b) and $f'' \in L_1(a,b)$. Then we have the inequality:*

$$\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \tag{2.1}$$

$$\leq \frac{1}{2(b-a)}\left(\left|x - \frac{a+b}{2}\right| + \frac{1}{2}(b-a)\right)^2 \|f''\|_1 \leq \frac{b-a}{2}\|f''\|_1$$

for all $x \in [a,b]$.

Proof. Let us define the mapping $K(\cdot,\cdot) : [a,b]^2 \to \mathbf{R}$ given by

$$K(x,t) := \begin{cases} \dfrac{(t-a)^2}{2} & \text{if } t \in [a,x] \\ \dfrac{(t-b)^2}{2} & \text{if } t \in (x,b]. \end{cases}$$

Integrating by parts, we have successively

$$\begin{aligned}
\int_a^b K(x,t) f''(t)\,dt &= \int_a^x \frac{(t-a)^2}{2} f''(t)\,dt + \int_x^b \frac{(t-b)^2}{2} f''(t)\,dt \\
&= \frac{(t-a)^2}{2} f'(t)\Big|_a^x - \int_a^x (t-a) f'(t)\,dt \\
&\quad + \frac{(t-b)^2}{2} f'(t)\Big|_x^b - \int_x^b (t-b) f'(t)\,dt \\
&= \frac{(x-a)^2}{2} f'(x) - \left[(t-a) f(t)\Big|_a^x - \int_a^x f(t)\,dt\right] \\
&\quad - \frac{(b-x)^2}{2} f'(x) - \left[(t-b) f(t)\Big|_x^b - \int_x^b f(t)\,dt\right] \\
&= \frac{1}{2}\left[(x-a)^2 - (b-x)^2\right] f'(x) - (x-a) f(x) \\
&\quad + \int_a^x f(t)\,dt + (x-b) f(x) + \int_x^b f(t)\,dt \\
&= (b-a)\left(x - \frac{a+b}{2}\right) f'(x) - (b-a) f(x) + \int_a^b f(t)\,dt
\end{aligned}$$

from where we get the integral identity:

$$\begin{aligned}
\int_a^b f(t)\,dt &= (b-a) f(x) - (b-a)\left(x - \frac{a+b}{2}\right) f'(x) \\
&\quad + \int_a^b K(x,t) f''(t)\,dt
\end{aligned} \tag{2.2}$$

for all $x \in [a,b]$, which is interesting in itself, too.

Using the identity (2.2) we have

$$\begin{aligned}
&\left| f(x) - \frac{1}{b-a}\int_a^b f(t)\,dt - \left(x - \frac{a+b}{2}\right) f'(x) \right| \\
&= \frac{1}{b-a}\left| \int_a^b K(x,t)\, f''(t)\,dt \right| \\
&= \frac{1}{b-a}\left| \int_a^x \frac{(t-a)^2}{2} f''(t)\,dt + \int_x^b \frac{(t-b)^2}{2} f''(t)\,dt \right| \\
&\le \frac{1}{b-a}\left[\int_a^x \frac{(t-a)^2}{2} \left|f''(t)\right| dt + \int_x^b \frac{(t-b)^2}{2} \left|f''(t)\right| dt \right] \\
&\le \frac{1}{b-a}\left[\frac{(x-a)^2}{2} \int_a^x \left|f''(t)\right| dt + \frac{(b-x)^2}{2} \int_x^b \left|f''(t)\right| dt \right] \\
&\le \frac{1}{b-a} \max\left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\} \\
&\quad \times \left[\int_a^x \left|f''(t)\right| dt + \int_x^b \left|f''(t)\right| dt \right]. \qquad (2.3)
\end{aligned}$$

Now, let observe that

$$\begin{aligned}
&\max\left\{ \frac{(x-a)^2}{2}, \frac{(b-x)^2}{2} \right\} \\
&= \frac{1}{2}\, \frac{(x-a)^2 + (b-x)^2 + \left|(b-x)^2 - (x-a)^2\right|}{2} \\
&= \frac{1}{2}\left[\frac{(x-a)^2 + (b-x)^2}{2} + (b-a)\left| x - \frac{a+b}{2} \right| \right] \\
&= \frac{1}{2}\left[\frac{1}{4}(b-a)^2 + \left(x - \frac{a+b}{2}\right)^2 + (b-a)\left| x - \frac{a+b}{2} \right| \right] \\
&= \frac{1}{2}\left(\left| x - \frac{a+b}{2} \right| + \frac{1}{2}(b-a) \right)^2.
\end{aligned}$$

Using (2.3) we deduce the desired inequality (2.1).

COROLLARY 5. *Let f be as above. Then we have the mid-point inequality:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right|\leq\frac{1}{8}(b-a)\|f''\|_1. \tag{2.4}$$

The following trapezoid inequality also holds:

COROLLARY 6. *Under the above assumptions we have:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{4}(f'(b)-f'(a))\right| \tag{2.5}$$

$$\leq\frac{1}{2}(b-a)\|f''\|_1.$$

Proof. Choose in (2.1) $x=a$ and $x=b$ to get:

$$\left|f(a)-\frac{1}{b-a}\int_a^b f(t)\,dt+\frac{b-a}{2}f'(a)\right|\leq\frac{b-a}{2}\|f''\|_1$$

and

$$\left|f(b)-\frac{1}{b-a}\int_a^b f(t)\,dt-\frac{b-a}{2}f'(b)\right|\leq\frac{b-a}{2}\|f''\|_1.$$

Adding the above two inequalities, using the triangle inequality and dividing by 2, we get the desired inequality (2.5).

3. Applications in Numerical Integration

Let $I_n: a=x_0<x_1<\ldots<x_{n-1}<x_n=b$ be a division of the interval $[a,b]$, $\xi_i\in[x_i,x_{i+1}]$ $(i=0,\cdots,n-1)$. We have the following quadrature formula:

THEOREM 7. *Let $f : [a,b] \to \mathbf{R}$ be continuous on $[a,b]$ and twice differentiable on (a,b), whose second derivative $f'' : (a,b) \to \mathbf{R}$ belongs to $L_1(a,b)$, i.e., $\|f''\|_1 := \int_a^b |f''(t)|\,dt < \infty$. Then the following perturbed Riemann's type quadrature formula holds:*

$$\int_a^b f(x)\,dx = A(f, f', \xi, I_n) + R(f, f', \xi, I_n) \tag{3.1}$$

where

$$A(f, f', \xi, I_n) = \sum_{i=0}^{n-1} h_i f(\xi_i) - \sum_{i=0}^{n-1} f'(\xi_i)\left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder satisfies the estimation:
(3.2)

$$|R(f, f', \xi, I_n)| \le \frac{1}{2}\left[\frac{1}{2}\nu(h) + \sup_{i=0,\cdots,n-1}\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right|\right]^2 \|f''\|_1$$

$$\le \frac{\nu^2(h)}{2}\|f''\|_1$$

for all ξ_i as above, where $\nu(h) = \max\{x_{i+1} - x_i | i = 0, \cdots, n-1\}$.

Proof. Apply Theorem 4 on the interval $[x_i, x_{i+1}]$ $(i = 0, \cdots, n-1)$ to get

$$\left|\int_{x_i}^{x_{i+1}} f(t)\,dt - h_i f(\xi_i) + \left(\xi_i - \frac{x_i + x_{i+1}}{2}\right) f'(\xi_i)\right|$$
$$\le \frac{1}{2}\left(\left|\xi_i - \frac{x_i + x_{i+1}}{2}\right| + \frac{1}{2}(x_{i+1} - x_i)\right)^2 \int_{x_i}^{x_{i+1}} |f''(t)|\,dt.$$

Summing over i from 0 to $n-1$ and using the generalized triangle inequality we deduce :

$$\begin{aligned}
|R(f,f',\xi,I_n)| &\leq \frac{1}{2}\sum_{i=0}^{n-1}\left[\frac{1}{2}(x_{i+1}-x_i)+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2 \\
&\quad\times\int_{x_i}^{x_{i+1}}|f''(t)|\,dt \\
&\leq \frac{1}{2}\sup_{i=0,\cdots,n-1}\left[\frac{1}{2}(x_{i+1}-x_i)+\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2 \\
&\quad\times\sum_{i=0}^{n-1}\int_{x_i}^{x_{i+1}}|f''(t)|\,dt \\
&\leq \frac{1}{2}\left[\frac{1}{2}\nu(h)+\sup_{i=0,\cdots,n-1}\left|\xi_i-\frac{x_i+x_{i+1}}{2}\right|\right]^2\|f''\|_1
\end{aligned}$$

and the estimation (3.2) is obtained.

REMARK 1. If we choose above $\xi_i=\frac{x_i+x_{i+1}}{2}$, we recapture the midpoint quadrature formula

$$\int_a^b f(x)\,dx = A_M(f,I_n)+R_M(f,I_n)$$

where the remainder $R_M(f,I_n)$ satisfies the estimation

$$|R_M(f,I_n)| \leq \frac{1}{8}\nu^2(h)\|f''\|_1 .$$

4. Applications for Special Means

Let us recall the following means :

(a) The arithmetic mean

$$A=A(a,b):=\frac{a+b}{2}, \qquad a,b\geq 0;$$

(b) The geometric mean:

$$G=G(a,b):=\sqrt{ab}, \qquad a,b\geq 0;$$

(c) The harmonic mean:

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \qquad a,b \geq 0;$$

(d) The logarithmic mean:

$$L = L(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } a \neq b \end{cases} \qquad a,b > 0;$$

(e) The identric mean:

$$I = I(a,b) := \begin{cases} a & \text{if } a = b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } a \neq b \end{cases} \qquad a,b > 0;$$

(f) The p-logarithmic mean:

$$L_p = L_p(a,b) := \begin{cases} \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } a \neq b; \\ a & \text{if } a = b \end{cases}$$

where, $p \in \mathbf{R}\setminus\{-1,0\}, a,b > 0$.

The following simple relationships are known in the literature

$$H \leq G \leq L \leq I \leq A.$$

It is also known that L_p is monotonically increasing in $p \in \mathbf{R}$ with $L_0 = I$ and $L_{-1} = L$.

EXAMPLE 1. Consider the mapping $f : (0,\infty) \to \mathbf{R}$, $f(x) = x^r$, $r \in \mathbf{R}\setminus\{-1,0\}$. Then we have for $0 < a < b$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L_r^r(a,b)$$

and

$$\|f''\|_1 = |r(r-1)|(b-a)L_{r-1}^{r-1}(a,b).$$

Using the inequality (2.1) we get :

$$(4.1)\qquad \left|x^r - L_r^r - r(x-A)x^{r-1}\right| \le \frac{1}{2}\left[|x-A| + \frac{1}{2}(b-a)\right]^2 |r(r-1)| L_{r-1}^{r-1}$$

for all $x \in [a,b]$. If in (4.1 we choose $x = A$, we get

$$(4.2)\qquad |A^r - L_r^r| \le \frac{|r(r-1)|(b-a)^2}{8} L_{r-1}^{r-1}.$$

EXAMPLE 2. Consider the mapping $f : (0,\infty) \to \mathbf{R}$, $f(x) = \frac{1}{x}$. Then we have for $0 < a < b$:

$$\frac{1}{b-a}\int_a^b f(x)\,dx = L^{-1}(a,b)$$

and

$$\|f''\|_1 = 2(b-a)L_{-3}^{-3}(a,b).$$

Using the inequality (2.1), we get :

$$\left|\frac{1}{x} - \frac{1}{L} + \frac{x-A}{x^2}\right| \le L_{-3}^{-3}\left[|x-A| + \frac{1}{2}(b-a)\right]^2$$

which is equivalent to

$$(4.3)\qquad |x(L-x) + L(x-A)| \le x^2 L L_{-3}^{-3}\left[|x-A| + \frac{1}{2}(b-a)\right]^2$$

for all $x \in [a,b]$. Now, if we choose in (4.3), $x = A$, we get

$$(4.4)\qquad 0 \le A - L \le \frac{1}{4} A L L_{-3}^{-3}(b-a)^2.$$

If in (4.3) we choose $x = L$, we get

$$(4.5)\qquad 0 \le A - L \le L^2 L_{-3}^{-3}\left[L - A + \frac{1}{2}(b-a)\right]^2.$$

EXAMPLE 3. Let us consider the mapping $f(x) = \ln x$, $x \in [a,b] \subset (0,\infty)$. Then we have :

$$\frac{1}{b-a}\int_a^b f(x)\,dx = \ln I(a,b),$$

and

$$\|f''\|_1 = (b-a)\,L_{-2}^{-2}(a,b).$$

Then the inequality (2.1) gives us

$$\left|\ln x - \ln I - \frac{x-A}{x}\right| \le \frac{1}{2}\left[|x-A| + \frac{1}{2}(b-a)\right]^2 L_{-2}^{-2} \tag{4.6}$$

for all $x \in [a,b]$.

Now, if in (4.6) we choose $x = A$, we get

$$1 \le \frac{A}{I} \le \exp\left[\frac{1}{8}(b-a)^2 L_{-2}^{-2}\right]. \tag{4.7}$$

If in (4.6) we choose $x = I$, we get

$$0 \le A - I \le \frac{I}{2}\left[A - I + \frac{1}{2}(b-a)\right]^2 L_{-2}^{-2}. \tag{4.8}$$

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