Journal of the Korea Society of Mathematical Education Series D:
Research in Mathematical Education
Vol. 2, No. 1 July 1998, 1–4

AN ELEMENTARY WAY OF ADDING TWO CANTOR SETS

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Let C be the Cantor set. It is well known that $C+C = \{x+y : x \in C, y \in C\} = [0,2]$ and $C - C = [-1,1]$.
We introduce a fairly elementary method for the proof which also works even for generalized Cantor sets.

1. Adding two Cantor sets

Let C be the Cantor set. Then we have $C + C = [0,2]$ and $C - C = [-1,1]$. A well-known proof for this uses ternary expression(see e.g. Rudin(1976), p. 81). More precisely, this can be seen by adding and subtracting members of C in base 3 recalling that C is the set of all real numbers in $[0,1]$ with only 0's and 2's in their ternary representations. Here we introduce a fairly elementary method which also works even for generalized Cantor sets.

Let I_n be the set appearing in the construction of the Cantor set C, that is,

$$
\begin{aligned}
I_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right], \\
I_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right], \\
&\cdots, etc.
\end{aligned}
$$

Then $C = \underset{n=1}{\rightarrow} \overset{\infty}{\rightarrow} \cap I_n$.

Consider the function $f(x,y) = x + y$, which is the projection of the plane onto the real axis along the line $x + y = 0$ (Figure 1).

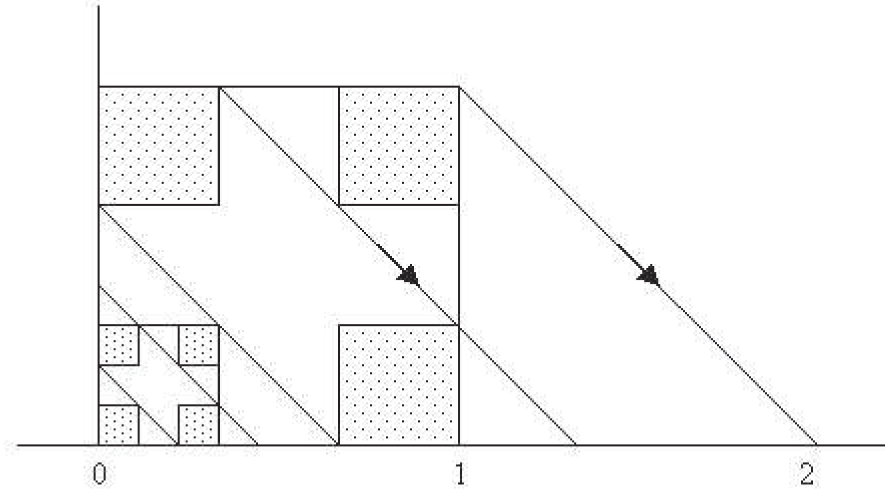

Figure 1. Projection along $x+y=0$

We see from Figure 1 that $f(I_1 \times I_1) = [0,2]$ and, by induction, that

$$f(I_n \times I_n) = [0,2] \text{ for } n = 1,2,3,\cdots .$$

Now we have

$$\begin{aligned}
[0,\ 2] \supseteq C + C &= f(C \times C) \\
&= f\left(\underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow}\cap I_n \times \underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow}\cap I_n\right) \\
&\supseteq f\left(\underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow}\cap (I_n \times I_n)\right) \\
&= \underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow}\cap f(I_n \times I_n) \\
&= \underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow}\cap [0,2] = [0,2].
\end{aligned}$$

Therefore $C + C = [0,\ 2]$.

Similarly, considering the function $g(x,y) = x - y$, which is the projection of the plane along the line $x - y = 0$ (Figure 2), we have $C - C = [-1,1]$.

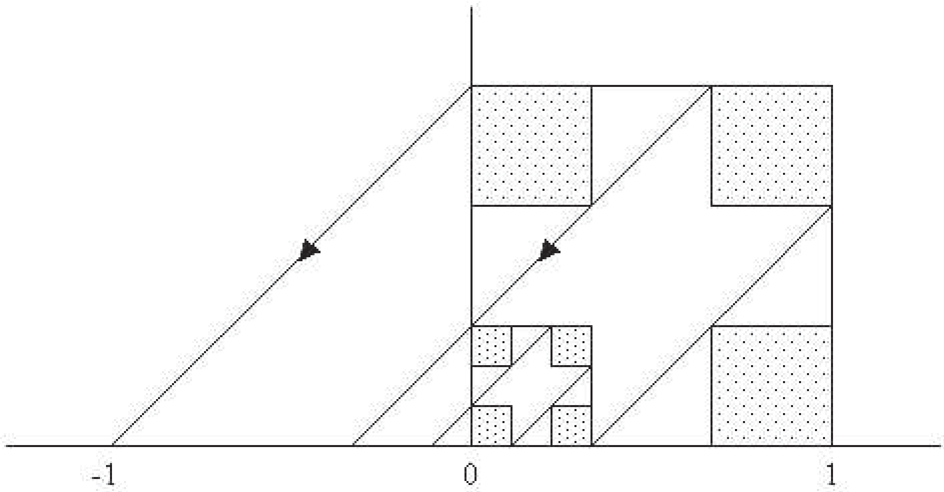

Figure 2. Projection along $x - y = 0$

2. Adding two generalized Cantor sets

The above method can be applied to the case of generalized Cantor sets. For $0 < \alpha < 1$, let C_α be the generalized Cantor set. (The deleted middle interval has length α times the length of the interval, e.g. $C_{\frac{1}{3}} = C$.) Note that there is no inclusion between C_α and C_β if $\alpha \neq \beta$.

Proposition.

(i) $C_\alpha + C_\alpha = [0, 2]$ *if* $0 < \alpha \leq \frac{1}{3}$

(ii) $C_\alpha + C_\alpha = \underset{n=1}{\rightarrow}\overset{\infty}{\rightarrow} \cap J_n$ *if* $\frac{1}{3} < \alpha < 1$, *where*

$$J_1 = [0, 1 - \alpha] \cup \left[\frac{1 + \alpha}{2}, \frac{3 - \alpha}{2}\right] \cup [1 + \alpha, 2]$$

and, for $n \geq 2$, J_n *is the set obtained by deleting two middle sets from each interval of* J_{n-1} *(Figure 3).*

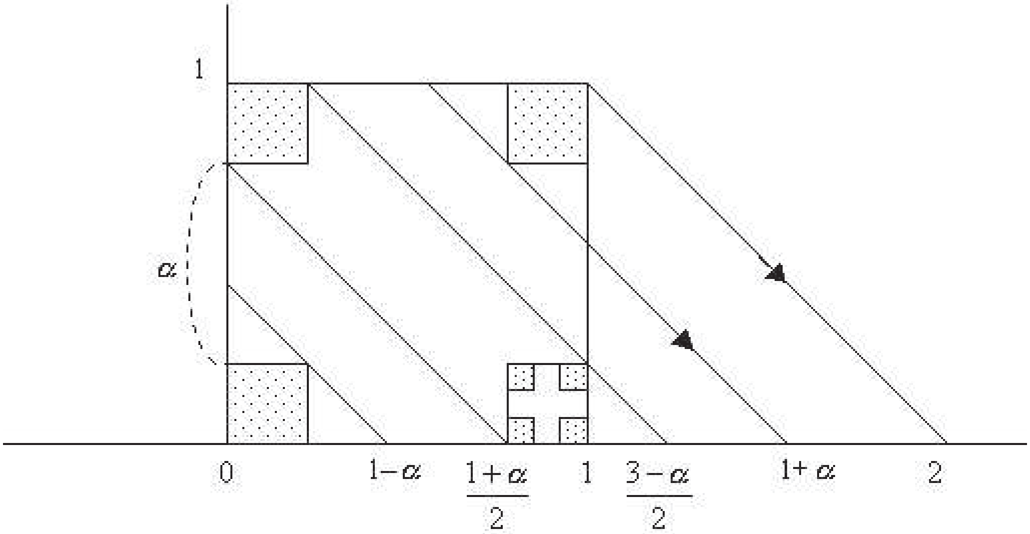

Figure 3. $[0, 1-\alpha] \cup \left[\frac{1+\alpha}{2}, \frac{3-\alpha}{2}\right] \cup [1+\alpha, 2]$

Remark 1. Similarly, we see that for any α the set $C_\alpha - C_\alpha$ is the mirror image of $C_\alpha + C_\alpha$ with respect to the point $x = \frac{1}{2}$.

Remark 2. As α passes by $\frac{1}{3}$, the measure of the set $C_\alpha \pm C_\alpha$ drops from 2 to 0. But the fractal dimension (see e.g. Crownover(1995), p. 7)

$$f\dim(C_\alpha \pm C_\alpha) = \begin{cases} 1, & 0 < \alpha \le \frac{1}{3} \\ \frac{\ln 3}{\ln 2 - \ln(1-\alpha)}, & \frac{1}{3} \le \alpha < 1 \end{cases}$$

is, as expected, a continuous function of α.

REFERENCES

Crownover, R. M. (1995): *Introduction to Fractals and Chaos*, Jones and Bartlett Publisher.
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