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Optimal Designs of Partially Accelerated Life Tests for Weibull Distributions*

와이블 분포에서 부분가속수명시험의 최적설계

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Abstract

This paper considers two modes of partially accelerated life tests for items having Weibull lifetime distributions. In a use-to-acclerated mode each item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until a predetermined censoring time. In an accelerated-to-use mode each one is first run at accelerated condition and, if it does not fail for a specified time, then it is run at use condition. Maximum likelihood estimators of the parameters of the lifetime distribution at use condition, and the 'acceleration factor' are obtained. The stress change time for each mode is determined to minimize the asymptotic variance of the acceleration factor, and the two modes are compared. For selected values of the design parameters the optimum plans are obtained, and the effects of the incorrect pre-estimates of the design parameters are investigated. Minimizing the generalized asymptotic variance of the estimators of the model parameters is also considered as an optimality criterion.

1. Introduction

When life testing of items at the specified use condition requires a long time to acquire the test data, accelerated life tests (ALTs) or partially accelerated life tests (PALTs) are often used to shorten the lives of test items. In an ALT test items are run only at higher-than-usual levels of stress, and in a PALT at both accelerated and use conditions. The test data obtained at the accelerated conditions are analyzed in terms of a model, and then extrapolated to the specified design stress to estimate the life distribution.

Nelson [18] gives various methods of estimating the parameters from ALT data. When lifetimes of items follow a Weibull or lognormal distribution, optimal constant stress ALT plans which minimize the asymptotic variance of maximum likelihood estimator (MLE) of a specified percentile at use condition are available in the

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literature; see Meeker and Hahn [14] and Nelson [18], and references therein. Nelson [17] presented a statistical model for analyzing step stress test data. For items having exponentially distributed lifetimes, Miller and Nelson [15], Bai et al. [6], and Bai and Chun [1] obtained optimum step stress ALTs which minimize the asymptotic variance of MLE of the log mean life at design stress. For items with Weibull lifetimes Schatzoff and Lane [20] studied the problem of determining stress change time with readout data, and Bai and Kim [4] presented step stress ALTs which minimize the asymptotic variance of MLE of a specified percentile at design stress. Bai and Kim [5] compared the step stress tests with the constant stress tests.

DeGroot and Goel [10] considered a PALT in which a test item is first run at use condition and, if it does not fail for a specified time τ, then it is run at accelerated condition until failure. They suggested a model in which $Y = T$ if $T \le \tau$, and $Y = \tau + (T - \tau)/\beta$ if $T > \tau$, where T is the lifetime of an item at use condition and Y its total lifetime. Assuming that T follows an exponential distribution with mean life α and using a Bayesian approach, they obtained estimators of β and α, and the optimal change time τ^*. Bhattacharyya and Soejoeti [7] proposed a failure rate model in which, if $h_T(\cdot)$ and $h_Y(\cdot)$ are the failure rate functions of T and Y, respectively, then $h_Y(y) = h_T(y)$ if $y \le \tau$ and $h_Y(y) = \beta h_T(y)$ if $y > \tau$. This model is equivalent to the DeGroot-Goel model when T has an exponential distribution.

An optimally designed PALT can have some practical usage when one wants to know the acceleration factor in order to carry out the test only at specified accelerated condition and to extrapolate the data to estimate the lifetime distribution at use condition. See DeGroot and Goel [10], and Bhattacharyya and Soejoeti [7]. The problem of optimally designing PALTs has been considered by DeGroot and Goel [10] and Bai and Chung [2] for items with exponentially distributed lifetimes, and by Bai et al. [3] for items having lognormally distributed lifetimes.

This paper considers designing two modes of PALTs, the use-to-accelerated (UA) and accelerated-to-use (AU) tests. In a UA mode, each test item is first run at use condition and, if it does not fail for a specified time, then it is run at accelerated condition until a predetermined censoring time. In a AU mode, however, each one is first run at accelerated condition and, if it does not fail for a specified time, then it is run at use condition. The AU mode can be adopted in the life tests using the alternate stress loading, thermal cycling (MIL-STD-781D [16] and Nelson [18]), etc. For items having Weibull lifetime distributions, the shape and scale parameters of the lifetime distribution at use condition, and the 'acceleration factor', which is defined as the ratio of the scale parameter of the use-condition distribution to that of the accelerated-condition distribution, are estimated by the method of maximum likelihood. The stress change time for each mode is determined to minimize the asymptotic variance of the acceleration factor. For selected combinations of the design parameters the optimum plans are obtained, and the effects of using the incorrect pre-estimates of the design parameters on the variances are investigated. The generalized asymptotic variance of MLEs of the model parameters is also considered as an optimality criterion.

2. The Model

Notation

n	number of test items
T_u, T_a	lifetimes of an item tested only at use and accelerated conditions
η	censoring time
τ_u, τ_a	change times from use to accelerated condition and from accelerated to use condition
x_u, x_a	standardized change times; $x_u = \tau_u/\eta, x_a = \tau_a/\eta$
Y_u, Y_a	lifetimes of an item tested by UA and AU modes

δ, α	shape and scale parameters of the Weibull distribution
β	acceleration factor (>1)
σ, μ, θ	$\sigma = 1/\delta$, $\mu = \ln\alpha$, $\theta = -\ln\beta$
$\phi(\cdot)$, $\Phi(\cdot)$	standard extreme value probability density function (p.d.f.) and cumulative distribution function (c.d.f.); $\phi(z) = e^{z-e^z}$, $\Phi(z) = 1 - e^{-e^z}$
$\Phi^{-1}(\cdot)$	inverse function of standard extreme value c.d.f.; $\Phi^{-1}(p) = \ln(-\ln(1-p))$
p_u, p_a	propabilities that an item tested only at use and accelerated conditions fail by η; $p_a = \Phi((\ln\eta - \mu - \theta)/\sigma)$, $p_u = \Phi((\ln\eta - \mu)/\sigma)$

2.1 UA PALT

Test procedure

1. Each of n test items is first run at use condition.
2. If it does not fail at use condition by τ_u, then it is put on accelerated condition and run until censoring time η.

Assumptions

1. T_u follows a Weibull distribution with parameters α and δ, i.e., $\Pr\{T_u \le t\} = 1 - e^{-(t/\alpha)^\delta} = \Phi((\ln t - \mu)/\sigma)$.
2. $Y_u = T_u$ if $T_u \le \tau_u$ and $Y_u = \tau_u + (T_u - \tau_u)/\beta$ if $T_u > \tau_u$.
3. The lifetimes of test items are statistically independent.

Lifetime distribution

Let $z_0(y) = (\ln y - \mu)/\sigma$, and $z_1(y) = \{\ln(y - \tau_u + \kappa_u) - \mu - \theta)\}/\sigma$, where $\kappa_u = \tau_u e^{\theta}$. From the above assumptions the p.d.f. of Y_u is;

$$f_u(y) = \begin{cases} 0, & y \le 0, \\ \phi(z_0(y))/(\sigma y), & 0 < y \le \tau_u, \\ \phi(z_1(y))/(\sigma(y - \tau_u + \kappa_u)), & \tau_u < y, \end{cases} \tag{2.1}$$

Estimation of parameters and the Fisher information matrix

The method of maximum likelihood is used to estimate parameters θ (or β), μ and σ from the test data. The change time τ_u is determined to minimize the asymptotic variance of MLE of β. The lifetimes $Y_{u1}, \cdots, Y_{un}$ of n test items are independent and identically distributed random variables. Let y_{ui} be the observed value of the lifetime of item i, and $D_{u0} = \{y_{ui} : 0 < y_{ui} \le \tau_u\}$, and $D_{u1} = \{y_{ui} : \tau_u < y_{ui} \le \eta\}$. And let indicator function I_{ij}^u be defined as

$$I_{ij}^u = I_{ij}^u(y_{ui}) = \begin{cases} 1, y_{ui} \in D_{uj} \\ 0, y_{ui} \notin D_{uj} \end{cases}, i = 1, 2, \cdots, n, j = 0, 1. \tag{2.2}$$

The log likelihood for $(y_{ui}, I_{i0}^u, I_{i1}^u)$ is

$$l_i^u(\theta, \mu, \sigma; y_{ui}, I_{i0}^u, I_{i1}^u) = I_{i0}^u\{-\ln\sigma - \ln y_{ui} + z_0(y_{ui}) - e^{z_0(y_{ui})}\} + I_{i1}^u\{-\ln\sigma - \ln(y_{ui} - \tau_u + \kappa_u) + z_1(y_{ui}) - e^{z_1(y_{ui})}\} - I_{i2}^u e^{z_1(\eta)}, \tag{2.3}$$

where $I_{i2}^u = 1 - I_{i0}^u - I_{i1}^u$. The total log likelihood for $(y_{u1}, I_{10}^u, I_{11}^u, \cdots, y_{un}, I_{n0}^u, I_{n1}^u)$ is $l^u(\theta, \mu, \sigma) = \sum_{i=1}^{n} l_i^u$.

MLEs $\hat\theta$, $\hat\mu$, and $\hat\sigma$ are the values of θ, μ and σ which are a solution to the system of equations obtained by letting the first partial derivatives of $l^u(\theta, \mu, \sigma)$ with respect to θ, μ and σ be zero. If we let $w(y_{ui}) = (y_{ui} - \tau_u)/(y_{ui} - \tau_u + \kappa_u)$, then the system of equations is as follows :

$$\sigma(\partial l^u/\partial\theta) = \sum_{i=1}^{n}[I_{i1}^u\{(\sigma - 1)w(y_{ui}) - \sigma + w(y_{ui})e^{z_1(y_{ui})}\} + I_{i2}^u w(\eta)e^{z_1(\eta)}] = 0 \tag{2.4}$$

$$\sigma(\partial l^u/\partial \mu) = \sum_{i=1}^{n}[I_{i0}^u\{-1+e^{z_0(y_{ui})}\}+I_{i1}^u\{-1+e^{z_1(y_{ui})}\} + I_{i2}^u e^{z_1(\eta)}]=0 \quad (2.5)$$

$$\sigma(\partial l^u/\partial \sigma) = \sum_{i=1}^{n}[I_{i0}^u\{-1-z_0(y_{ui})+z_0(y_{ui})e^{z_0(y_{ui})}\}+ I_{i1}^u\{-1-z_1(y_{ui})+z_1(y_{ui})e^{z_1(y_{ui})}\}+ I_{i2}^u z_1(\eta)e^{z_1(\eta)}]=0 \quad (2.6)$$

The solution to the above system of equations can be obtained by using a numerical method such as Newton-Raphson method. The Fisher information matrix is obtained by taking expectations of the negative second partial derivatives of $l^u(\theta, \mu, \sigma)$ with respect to θ, μ and σ. See Appendices for details.

2.2 AU PALT

Test procedure

1. Each of n test items is first run at accelerated condition.
2. If it does not fail at accelerated condition by τ_a, then it is put on use condition and run until censoring time η.

Assumptions

1. $T_a = T_u/\beta$ follows a Weibull distribution with parameters α/β and δ, i.e., $\Pr[T_a \le t] = \Phi((\ln t-\mu-\theta)/\sigma)$.
2. $Y_a = T_a$ if $T_a \le \tau_a$ and $Y_a = \tau_a + (T_a - \tau_a)\beta$ if $T_a > \tau_a$.
3. The lifetimes of test items are statistically independent.

Lifetime distribution and estimation of parameters

From the above assumptions the p.d.f. of Y_a is

$$f_a(y)=\begin{cases} 0, & y \le 0, \\ \phi(z_0(y))/(\sigma y), & 0 < y \le \tau_a, \\ \phi(z_1(y))/(\sigma(y-\tau_a+x_a)), & \tau_a < y, \end{cases} \quad (2.7)$$

where $x_a = \tau_a e^{-\theta}$, $z_0(y)=(\ln y-\mu-\theta)/\sigma$, $z_1(y)=\{\ln(y-\tau_a+x_a)-\mu\}/\sigma$. Let y_{ai} be the observed value of the lifetime of item i, $D_{a0}=\{y_{ai}|0<y_{ai}\le\tau_a\}$ and $D_{a1}=\{y_{ai}|\tau_a<y_{ai}\le\eta\}$, and I_{ij}^a be the indicator function defined as

$$I_{ij}^a=I_{ij}^a(y_{ai}) = \begin{cases} 1, y_{ai}\in D_{aj} \\ 0, y_{ai}\notin D_{aj} \end{cases}, i=1,2,\cdots,n, j=0,1.$$

The log likelihood for $(y_{ai}, I_{i1}^a, I_{i2}^a)$ is

$$l_i^a = I_{i0}^a\{-\ln\sigma-\ln y_{ai}+z_0(y_{ai})-e^{z_0(y_{ai})}\} +I_{i1}^a\{-\ln\sigma-\ln(y_{ai}-\tau_a+x_a)+z_1(y_{ai})-e^{z_1(y_{ai})}\}-I_{i2}^a e^{z_1(\eta)}. \quad (2.8)$$

where $I_{i2}^a=1-I_{i0}^a-I_{i1}^a$. MLEs $\hat\theta$, $\hat\mu$, and $\hat\sigma$ are the values of θ, μ and σ which are a solution to the system of equations obtained by letting the first partial derivatives of $l^a(\theta,\mu,\sigma)=\sum_{i=1}^{n} l_i^a$ with respect to θ, μ and σ be zero.

2.3 Asymptotic variance of MLE of the acceleration factor

If a reliable estimate of the acceleration factor can be obtained from past and/or current data, then tests may be conducted only at accelerated condition, and using the estimate one can extrapolate the test data into use condition provided that the production process from which the test items are sampled is stable for a period of time. In such a situation precision in the estimate of the acceleration factor would be of primary interest, and minimizing the asymptotic variance (Asvar) of MLE $\hat\beta$ of β would be a natural criterion.

The asymptotic variance-covariance matrix of MLEs $\hat\theta$, $\hat\mu$, and $\hat\sigma$ is the inverse matrix of the Fisher information matrix $F_n(\theta,\mu,\sigma)$ (see Appendices). Then

$$\text{Asvar}(\hat\beta) \bullet \beta^2 \cdot \text{Asvar}(\hat\theta), \quad (2.9)$$

and minimizing Asvar($\hat\beta$) is equivalent to minimizing

Asvar($\hat{\theta}$). For the case without censoring Asvar($\hat{\theta}$) for UA mode is the same as Asvar($\hat{\theta}$) for AU mode, and Asvar($\hat{\theta}$) = $\frac{n^2\pi^2}{6\sigma^2}|F_n(\theta,\mu,\sigma)|^{-1}$, where $|F_n(\theta,\mu,\sigma)|$ is the determinant of matrix $F_n(\theta,\mu,\sigma)$ (see formula A.1 of Appendices for $|F_n(\theta,\mu,\sigma)|$).

3. Optimum Plans

Design parameters

The optimal stress change times depend on model parameters θ, μ and σ. A design using the pre-estimates of the unknown model parameters is called a locally optimal design (Chernoff [8, 9]), and is commonly adopted. Let $a = \Phi^{-1}(p_a) = (\ln\eta - \mu - \theta)/\sigma$, and $b = \Phi^{-1}(p_a) - \Phi^{-1}(p_u) = -\theta/\sigma = \ln\beta/\sigma$.

PALTs can be designed using the pre-estimate of either one of three sets(θ, μ, σ), (p_u,p_a, σ), or (a,b, σ). It can be easily shown that the Fisher information matrix and asymptotic variance (2.9) for each mode can be written in terms of x_u or x_a, and either of (θ, μ, σ), (p_u,p_a, σ), or (a,b, σ).

Optimal stress change times

The optimal stress change time x_u* or x_a* can be obtained numerically, for example, by the Powell [19] method. The failure rate function $\lambda_T(t)$ of the lifetime T of an item at use condition is equal to $(e^{\mu}\sigma)^{-1}(t/e^{\mu})^{\sigma^{-1}-1}$, and that $\lambda_T(t)$ i) is constant for $\sigma = 1$, ii) increases for $\sigma < 1$, and iii) decreases for $\sigma > 1$.

We have computed the optimal stress change times for various combinations of (a,b, σ) values. Values of a = -.5(1)2.5, b = .1 ~ 10, and σ = .8 are considered. Fig. 1 shows x_u* and 1 - x_a* minimizing Asvar($\hat{\theta}$), where the solid lines are for UA mode and the dashed lines for AU mode. Fig. 2 shows the ratios of the optimal nAsvar($\hat{\theta}$)'s for UA and AU modes. Note that for most combinations of the (a,b, σ) considered the optimal nAsvar($\hat{\theta}$) of AU mode is less than that of UA mode.

As an illustrative example we slightly modified the example of Bai et al. [4] so that the use and accelerated temperatures are 150℃ and 170℃, respectively, η = 8760

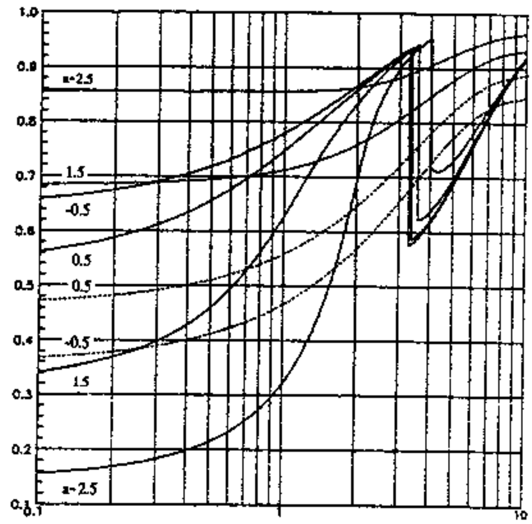

Fig. 1. Standardized change times x_u* & 1 - x_a* minimizing Asvar($\hat{\theta}$); σ = .8

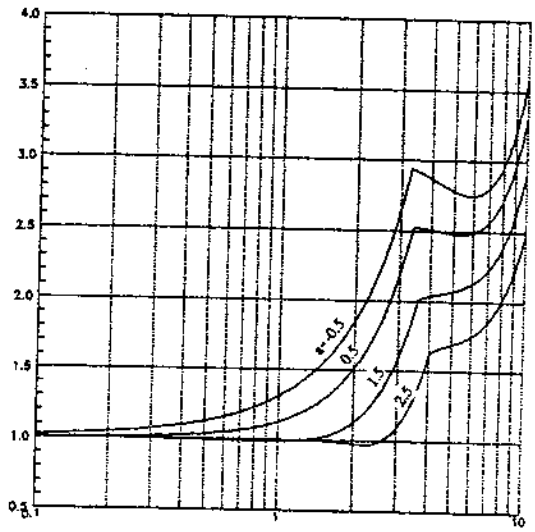

Fig. 2. Ratios of optimal nAsvar($\hat{\theta}$)'s for two modes; σ = .8

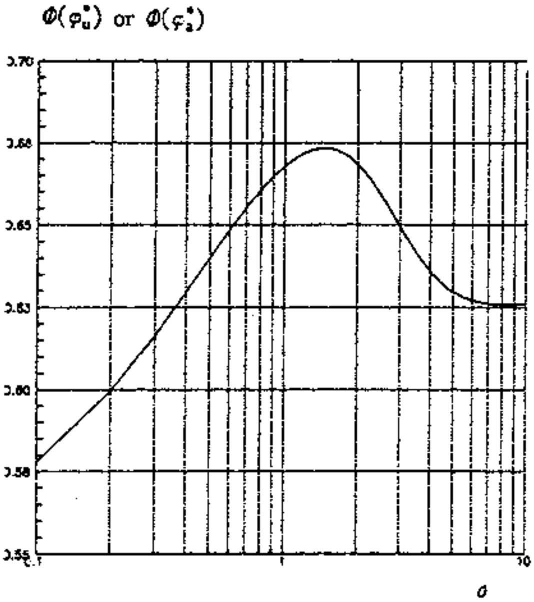

Fig. 3. $\Phi(\phi_u^*)$ and $\Phi(\phi_a^*)$ without censoring as a function of σ)

hours, and σ = .8 (δ = 1.25). Suppose that the pre-estimate of (a,b) is (1.5, 2), which is equivalent to (p_u,p_a) $\cong$(.45, .99), (θ, μ)$\cong$(-1.6, 9.48) or (α,β)$\cong$(13095, 4.95),. Then, Fig. 1 gives x_u^* = .84 (τ_u = 7358h) and x_a^* = 1 - .76 = .24 (τ_a = 2102h). We can see that the ratio of variances is 1.13 from Fig. 2, which means that the sample size of UA mode is 1.13 times larger than that of AU mode for the same asymptotic variance. In UA mode test items randomly chosen are first simultaneously run at 150℃ for 7385h, and the surviving items are then run at 170℃ until 8760h. In AU mode they are first run at 170℃ for 2102h, and the surviving items run at 170℃ until 8760h.

Case without censoring

Asvar($\hat{\beta}$) for the case without censoring depends on θ, μ, σ and τ_u or τ_a only through σ, and $\phi_u = (\ln \tau_u - \mu)/\sigma$ or $\phi_a = (\ln \tau_a - \mu - \theta)/\sigma$. Equivalently it depends on σ, and $\Phi(\phi_u)$ or $\Phi(\phi_a)$. Fig. 3 gives $\Phi(\phi_u^*)$ or $\Phi(\phi_a^*)$ as a function of σ. As an example suppose that the pre-estimate of (θ, μ, σ) is equal to (-2, 5, .8). Then, for UA mode the optimal stress change time is τ_u^* = 5 + .8 · Φ^{-1}(.66) = 5 + .8 · ln(-ln(1-.66)) = 5.06, and for AU

Table 1. PAVIs minimizing Asvar($\hat{\theta}$):(a,b, σ) = (2, 2, .8)

b'	σ' = 0.4			σ' = 0.8			σ' = 1.2		
	a'=1.8	2.0	2.2	1.8	2.0	2.2	1.8	2.0	2.2
1.6	.19[1]	.85	2.27	5.28	11.67	1.74	7.65	7.65	23.38
	312.80[2]	226.95	163.16	.41	.50	34.02	54.71	54.71	82.68
1.7	.02	.34	1.27	.91	2.82	7.28	.24	2.89	13.79
	307.79	224.12	161.66	3.21	.24	.67	38.38	59.40	87.28
1.8	.02	.07	.58	.20	1.19	3.98	.08	.50	6.12
	302.69	221.17	160.06	2.56	.11	.98	43.26	64.65	92.49
1.9	.19	.00	.17	.00	.28	1.75	1.05	.04	1.50
	297.52	218.12	158.36	1.97	.03	1.16	48.71	70.49	98.31
2.0	.49	.12	.01	.25	.00	.48	3.00	1.07	.01
	292.29	214.99	156.57	1.44	.00	1.49	54.75	76.93	104.75
2.1	.91	.40	.06	.91	.26	.01	5.88	3.32	.83
	287.04	211.78	154.70	.99	.03	1.87	61.40	83.98	111.83
2.2	1.44	.82	.30	1.95	.99	.21	9.66	6.64	3.30
	281.76	208.50	152.75	.62	.62	2.33	68.69	91.66	119.52
2.3	2.06	1.36	.71	3.33	2.14	.98	14.33	10.97	7.08
	276.48	205.17	150.73	.33	.30	2.86	76.63	99.97	127.82
2.4	2.76	2.01	1.25	5.05	3.69	2.25	19.92	16.30	12.02
	271.20	201.79	148.64	.13	.55	3.47	85.23	108.90	136.73

1) UA PALT 2) AU PALT

mode the optimal stress change time is $\tau_a^* = 5 + (-2) + .8 \cdot \ln(-\ln(1-.66)) = 3.06$.

Effects of incorrect pre-estimates

An optimal design is determined by specifying the value of (a,b, σ), which is usually unknown. Therefore, the value of (a,b, σ) has to be pre-estimated from past experience, data for example similar item, or a preliminary test. Incorrect pre-estimates give a non-optimal test. We investigate the effects of the incorrect pre-estimate of (a,b, σ) in terms of the percentage of the asymptotic variance increase (PAVI).

For the pre-estimate of (a,b, σ) = (2, 2, .8), Table 1 shows PAVIs due to using incorrect pre-estimate (a',b', σ') of (a,b, σ) for the test minimizing Asvar($\hat{\theta}$). Table 1 indicates that, except when correct pre-estimate of σ is used, PAVIs for UA mode seem to be smaller than PAVIs for AU mode.

4. The Generalized Asymptotic Variance

When estimation of the lifetime distribution at use condition is of primary interest, minimization of the generalized asymptotic variance of MLEs $\hat{\theta}$, $\hat{\mu}$, and $\hat{\sigma}$ would be a natural optimality criterion. The generalized asymptotic variance (GeAsvar) of ($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$) is

$$\text{GeAsvar}(\hat{\theta}, \hat{\mu}, \hat{\sigma}) = |F_n(\theta, \mu, \sigma)|^{-1}. \tag{4.1}$$

For the case without censoring GeAsvar ($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$)'s for the two modes are the same; see the Appendix.

For the same combinations of design parameters as Fig. 1, Fig. 4 shows x_u^* and $1 - x_a^*$ minimizing GeAsvar ($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$), and Fig. 5 shows the ratios of the optimal nGeAsvar ($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$)'s for UA and AU modes. Note that for the values of a = 2.5, b = 1 ~ 10 considered the optimal nAsvar($\hat{\theta}$) of UA mode is less than that of AU mode.

For the example of (a,b, σ) = (1.5, 2, .8) of Section 3

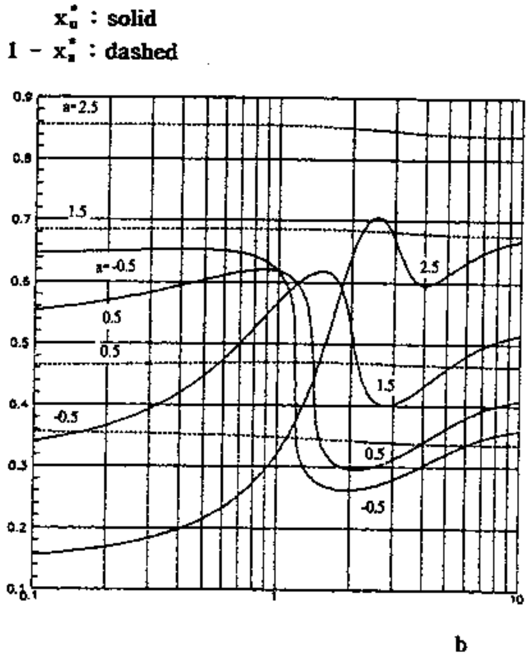

Fig. 4. Standardized change times x_u^* & $1 - x_a^*$ minimizing GeAsvar ($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$); σ =.8)

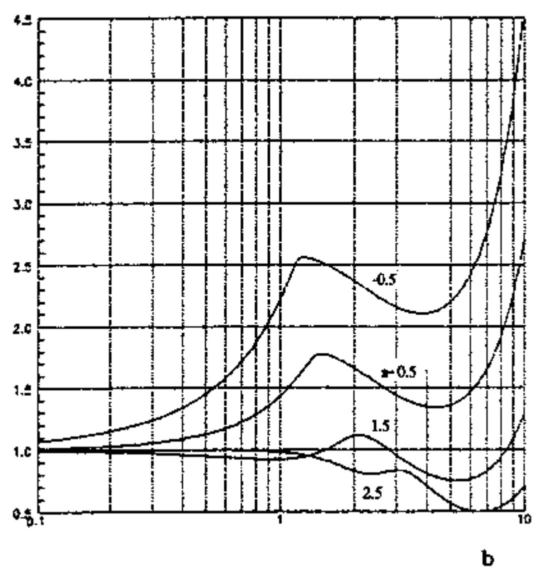

Fig. 5. Ratios of optimal nGeAsvar($\hat{\theta}$, $\hat{\mu}$, $\hat{\sigma}$)'s for the two modes; σ = .8)

the optimal stress change times minimizing GeAsvar $(\hat{\theta}, \hat{\mu}, \hat{\sigma})$ are $x_u^* = .53$ and $x_a^* = 1 - .68 = .32$ from Fig. 4, and the ratio of variances is 1.11 from Fig. 5. For the pre-estimate of $(a, b, \sigma) = (2, 2, .8)$ of Section 3 the test based on GeAsvar $(\hat{\theta}, \hat{\mu}, \hat{\sigma})$ exihibits patterns similar to the test based on Asvar$(\hat{\theta})$; when incorrect pre-estimate of σ is used, PAVIs for UA mode seem to be smaller than PAVIs for AU mode..

Effects of scale parameter, σ

Fig. 6 and 7 show, for each optimality criterion, the effects of the scale parameter, σ on the optimal stress change times for selected values of (a, b). The value of $1 - x_a^*$ increases as σ increase for each criterion, and the value of x_u^* decreases for the criterion of minimizing GeAsvar $(\hat{\theta}, \hat{\mu}, \hat{\sigma})$.

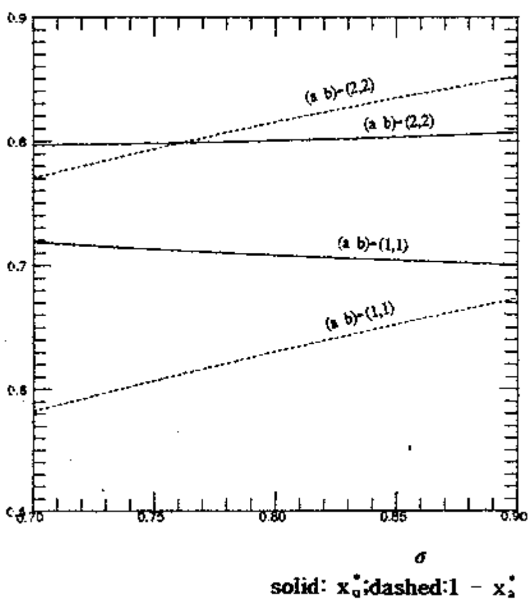

Fig. 6. Effects of σ on x_u^* & $1 - x_a^*$ minimizing Asvar$(\hat{\theta})$)

5. Concluding Remarks

We have considered the problem of optimally designing two modes of PALTs for items with Weibull-distributed lifetimes. Our results show that, when one can be free to choose between the two modes of PALTs, it may be natural, for the optimality criterion taken and the pre-estimted values of the design parameters, to compare the corresponding asymptotic variances and PAVIs of the optimal use-to-accelerated and accelerated-to-use testing modes.

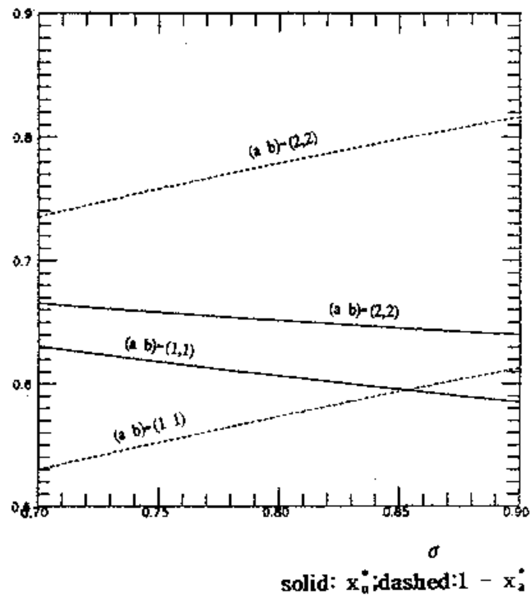

Fig. 7. Effects of σ on x_u^* & $1 - x_a^*$ minimizing GeAsvar $(\hat{\theta}, \hat{\mu}, \hat{\sigma})$)

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A. Appendices

A.1 Fisher information matrix for UA PALT

The second partial derivatives of l^u (2.3) with respect to θ, μ and σ are :

$$-\sigma^2(\partial^2 l^u/\partial\theta^2) = \sigma^2(\partial l^u/\partial\theta) + \sum_{i=1}^{n}\big[I_{i1}^u\{\sigma^2+\sigma(1-\sigma)w^2(y_{ui}) + (1-\sigma)w^2(y_{ui})e^{z_1(y_{ui})}\} + I_{i2}^u\{(1-\sigma)w^2(\eta)e^{z_1(\eta)}\}\big],$$

$$-\sigma^2(\partial^2 l^u/\partial\mu\partial\theta) = \sigma(\partial l^u/\partial\theta) + \sum_{i=1}^{n} I_{i1}^u\{(1-\sigma)w(y_{ui})+\sigma\},$$

$$-\sigma^2(\partial^2 l^u/\partial\sigma\partial\theta) = \sigma(\partial l^u/\partial\theta) + \sum_{i=1}^{n}\big[I_{i1}^u\{\sigma(1-w(y_{ui}))+w(y_{ui})z_1(y_{ui})e^{z_1(y_{ui})}\} + I_{i2}^u w(\eta)z_1(\eta)e^{z_1(\eta)}\big],$$

$$-\sigma^2(\partial^2 l^u/\partial\mu^2) = \sigma(\partial l^u/\partial\mu) + \sum_{i=1}^{n}(I_{i0}^u + I_{i1}^u),$$

$$-\sigma^2(\partial^2 l^u/\partial\sigma\partial\mu) = \sigma(\partial l^u/\partial\mu) + \sum_{i=1}^{n}\big[I_{i0}^u z_0(y_{ui})e^{z_0(y_{ui})} + I_{i1}^u z_1(y_{ui})e^{z_1(y_{ui})} + I_{i2}^u z_1(\eta)e^{z_1(\eta)}\big],$$

and

$$-\sigma^2(\partial^2 l^u/\partial\sigma^2) = 2\sigma(\partial l^u/\partial\sigma) + \sum_{i=1}^{n}\big[I_{i0}^u\{1+z_0^2(y_{ui})e^{z_0(y_{ui})}\} + I_{i1}^u\{1+z_1^2(y_{ui})e^{z_1(y_{ui})}\} + I_{i2}^u z_1^2(\eta)e^{z_1(\eta)}\big].$$

If we let $\varphi \equiv (\ln\tau_u - \mu)/\sigma$, and $\zeta \equiv z_1(\eta)$ the expectations needed for the Fisher information matrix are obtained as :

$$E[I_{i0}^u] = \int_{-\infty}^{\varphi}\phi(z)dz = \Phi(\varphi)$$

$$E[I_{i1}^u] = \int_{\varphi}^{\zeta}\phi(z)dz = \Phi(\zeta) - \Phi(\varphi)$$

$$E[I_{i2}^u] = 1 - \Phi(\zeta)$$

$$E[I_{i0}^u z_0^j(y_{ui})e^{z_0(y_{ui})}] = \int_0^{\tau_u}\left(\frac{\ln y-\mu}{\sigma}\right)^j e^{\frac{\ln y-\mu}{\sigma}} \cdot f_u(y)dy = \int_0^{e^{\varphi}}(\ln z)^j z e^{-z}dz,\ j = 1, 2$$

$$E[I_{i1}^{u} z_1^{j}(y_{ui}) e^{z_1(y_{ui})}] = \int_{\tau_u}^{\eta} \left(\frac{\ln(y-\tau_u+\chi_u)-\mu-\theta}{\sigma} \right)^{j} e^{\frac{\ln(y-\tau_u+\chi_u)-\mu-\theta}{\sigma}} \cdot f_u(y)dy$$

$$= \int_{e^{\varphi}}^{e^{\zeta}} (\ln z)^{j} z e^{-z} dz, \; j = 1, 2$$

$$E[I_{i1}^{u} w(y_{ui})] = \Phi(\zeta) - \Phi(\varphi) - e^{\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-\sigma-1}) z e^{-z} dz$$

$$E[I_{i1}^{u} w^2(y_{ui})] = \Phi(\zeta) - \Phi(\varphi) - 2e^{\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-\sigma-1}) z e^{-z} dz + e^{2\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-2\sigma-1}) z e^{-z} dz$$

$$E[I_{i1}^{u} w^2(y_{ui}) e^{z_1(y_{ui})}] = (1+e^{\varphi}) e^{-e^{\varphi}} - (1+e^{\zeta}) e^{-e^{\zeta}} + 2e^{\varphi\sigma} \Big\{ (\sigma-1) \int_{e^{\varphi}}^{e^{\zeta}} (z^{-\sigma-1}) z e^{-z} dz +$$

$$e^{-\zeta\sigma} e^{\zeta} e^{-e^{\zeta}} - e^{-\varphi\sigma} e^{\varphi} e^{-e^{\varphi}} \Big\} - e^{2\varphi\sigma} \Big\{ (2\sigma-1) \int_{e^{\varphi}}^{e^{\zeta}} (z^{-2\sigma-1}) z e^{-z} dz + e^{-2\zeta\sigma} e^{\zeta} e^{-e^{\zeta}} - e^{-2\varphi\sigma} e^{\varphi} e^{-e^{\varphi}} \Big\}$$

$$E[I_{i1}^{u} w(y_{ui}) z_1(y_{ui}) e^{z_1(y_{ui})}] = \int_{e^{\varphi}}^{e^{\zeta}} (\ln z) z e^{-z} dz - e^{\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-\sigma} \ln z) z e^{-z} dz.$$

The Fisher information matrix, obtained by taking expectations of the negative second partial derivatives, depends on φ, ζ and σ. φ and ζ can be expressed in terms of a, b, x_u and σ. That is, $\varphi = (1/\sigma)(\ln\eta-\mu) + (1/\sigma)\ln(\tau_u/\eta) = a - b + (1/\sigma)\ln x_u$, and $\zeta = (1/\sigma)(\ln\eta-\mu-\theta) + (1/\sigma)(\ln(1-\tau_u/\eta+(\tau_u/\eta)e^{\theta})) = a + (1/\sigma)(\ln(1-x_u+x_u e^{-b\sigma}))$ .We have evaluated the integrals involved using the algorithm of Escobar and Meeker [11, 12] and the DGAMIC, and DQDAG routines of IMSL [13].

For the case without censoring $I_{i2}^{u} = 0$, $I_{i0}^{u} + I_{i1}^{u} = 1$, $\lim_{\zeta-\infty} \Phi(\zeta) = 1$, and the upper limit of integrals is $e^{\zeta} \to \infty$.

The resulting expectations depend on φ and σ, or equivalently on $\Phi(\varphi)$ and σ. If we let $d_1 \equiv \int_{e^{\varphi}}^{\infty} (\ln z) z e^{-z} dz$, $d_2 \equiv \int_{e^{\varphi}}^{\infty} (z^{-\sigma-1}) z e^{-z} dz$, $d_3 \equiv \int_{e^{\varphi}}^{\infty} (z^{-2\sigma-1}) z e^{-z} dz$, and $d_4 \equiv \int_{e^{\varphi}}^{\infty} (z^{-\sigma} \ln z) z e^{-z} dz$, then the determinant of the Fisher information matrix is :

$$|F_n(\theta,\mu,\sigma)| = -(1-\gamma)^2 + \left(\frac{\pi^2}{6} + 2(1-\gamma)^2\right)\Phi(\varphi) - \left(\frac{\pi^2}{6} + (1-\gamma)^2\right)\Phi^2(\varphi)$$

$$+ 2(1-\gamma)d_1 + 2(1-\gamma)(1-\gamma+\sigma\gamma)e^{\varphi\sigma}d_2 + \frac{\pi^2}{6}(1-\sigma)^2 e^{2\varphi\sigma} d_3 - 2(1-\gamma)e^{\varphi\sigma}d_4$$

$$- 2(1-\gamma)\Phi(\varphi)d_1 - 2\left\{\sigma(1-\gamma) + (1-\sigma)\left(\frac{\pi^2}{6} + (1-\gamma)^2\right)\right\} e^{\varphi\sigma}\Phi(\varphi)d_2$$

$$+\ 2(1-\gamma)e^{\varphi\sigma}\Phi(\varphi)\dot{d}_4 - 2(1-\gamma+\sigma\gamma)e^{\varphi\sigma}d_1d_2 + 2e^{\varphi\sigma}d_1d_4 + 2(1-\gamma+\sigma\gamma)e^{\varphi\sigma}d_2d_4$$

$$- d_1^2 - 2\left\{\sigma(1-\sigma)(1-\gamma)+\sigma^2+(1-\sigma)^2\left(\frac{\pi^2}{6}+(1-\gamma)^2\right)\right\}e^{2\varphi\sigma}d_2^2 + 2e^{\varphi\sigma}d_4^2 \tag{7.1}$$

A.2 Fisher information matrix for AU PALT

The second partial derivatives of l^a with respect to θ, μ and σ are

$$-\sigma^2(\partial^2 l^a/\partial\theta^2) = \sigma^2(\partial l^a/\partial\theta) + \sum_{i=1}^{n}\Big[I_{i0}^a\{\sigma+(1-\sigma)e^{z_0(y_{ai})}\} +$$

$$I_{i1}^a\{\sigma(1-\sigma)w^2(y_{ai}) + (1-\sigma)w^2(y_{ai})e^{z_1(y_{ai})}\} + I_{i2}^a(1-\sigma)w^2(\eta)e^{z_1(\eta)}\Big],$$

$$-\sigma^2(\partial^2 l^a/\partial\mu\partial\theta) = \sigma(\partial l^a/\partial\theta) + \sum_{i=1}^{n}\Big[I_{i0}^a + I_{i1}^a(1-\sigma)w(y_{ai})\Big],$$

$$-\sigma^2(\partial^2 l^a/\partial\sigma\partial\theta) = \sigma(\partial l^a/\partial\theta) + \sum_{i=1}^{n}\Big[I_{i0}^a z_0(y_{ai})e^{z_0(y_{ai})} +$$

$$I_{i1}^a\{-\sigma w(y_{ai}) + w(y_{ai})z_1(y_{ai})e^{z_1(y_{ai})}\} + I_{i2}^a w(\eta)z_1(\eta)e^{z_1(\eta)}\Big],$$

$$-\sigma^2(\partial^2 l^a/\partial\mu^2) = \sigma(\partial l^a/\partial\mu) + \sum_{i=1}^{n}(I_{i0}^a + I_{i1}^a),$$

$$-\sigma^2(\partial^2 l^a/\partial\sigma\partial\mu) = \sigma(\partial l^a/\partial\mu) + \sum_{i=1}^{n}\Big[I_{i0}^a z_0(y_{ai})e^{z_0(y_{ai})} +$$

$$I_{i1}^a z_1(y_{ai})e^{z_1(y_{ai})} + I_{i2}^a z_1(\eta)e^{z_1(\eta)}\Big], \text{ and}$$

$$-\sigma^2(\partial^2 l^a/\partial\sigma^2) = 2\sigma(\partial l^a/\partial\sigma) + \sum_{i=1}^{n}\Big[I_{i0}^a\{1 + z_0^2(y_{ai})e^{z_0(y_{ai})}\} +$$

$$I_{i1}^a\{1 + z_1^2(y_{ai})e^{z_1(y_{ai})}\} + I_{i2}^a z_1^2(\eta)e^{z_1(\eta)}\Big],$$

where $w(y) = \chi_a/(y-\tau_a+\chi_a)$.

If we let $\varphi \equiv (\ln \tau_a - \mu - \theta)/\sigma$ and $\zeta \equiv z_1(\eta)$ the expectations needed for the Fisher information matrix are obtained as :

$$E[I_{i0}^a] = \Phi(\varphi)$$

$$E[I_{i1}^a] = \Phi(\zeta) - \Phi(\varphi)$$

$$E[I_{i2}^a] = 1 - \Phi(\zeta)$$

$$E[I_{i0}^a e^{z_0(y_{ai})}] = 1 - (1+e^{\varphi})e^{-e^{\varphi}},\ E[I_{i0}^a z_0^j(y_{ai}) e^{z_0(y_{ai})}] = \int_0^{e^{\varphi}} (\ln z)^j z e^{-z} dz,\ j = 1, 2$$

$$E[I_{i1}^a z_1^j(y_{ai}) e^{z_1(y_{ai})}] = \int_{e^{\varphi}}^{e^{\zeta}} (\ln z)^j z e^{-z} dz,\ E[I_{i1}^a w^j(y_{ai})] = e^{j\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-j\sigma-1}) z e^{-z} dz,\ j = 1, 2$$

$$E[I_{i1}^a w^2(y_{ai}) e^{z_1(y_{ai})}] = -e^{2\varphi\sigma}\left\{ (2\sigma-1) \int_{e^{\varphi}}^{e^{\zeta}} (z^{-2\sigma-1}) z e^{-z} dz + e^{-2\zeta\sigma} e^{\zeta} e^{-e^{\zeta}} - e^{-2\varphi\sigma} e^{\varphi} e^{-e^{\varphi}} \right\}$$

$$E[I_{i1}^a w(y_{ai}) z_1(y_{ai}) e^{z_1(y_{ai})}] = e^{\varphi\sigma} \int_{e^{\varphi}}^{e^{\zeta}} (z^{-\sigma} \ln z) z e^{-z} dz$$

These expectations are functions of φ, ζ and σ. φ and ζ can be expressed in terms of a, b, x_a and σ. That is, $\varphi = a + (1/\sigma) \ln x_a$, and $\zeta = a - b + (1/\sigma)(\ln(1 - x_a + x_a e^{b\sigma}))$. For the case without censoring $I_{i2}^a = 0$, $I_{i0}^a + I_{i1}^a = 1$, $\lim_{\zeta \to \infty} \Phi(\zeta) = 1$, and the upper limit of integrals is $e^{\zeta} \to \infty$. The resulting expectations depend on φ and σ, or equivalently on $\Phi(\varphi)$ and σ. Note that the determinant of the Fisher information matrix for AU mode is the same as that for UA mode, i.e., formula (7.1) with φ defined as $(\ln \tau_a - \mu - \theta)/\sigma$.