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SINGULAR INTEGRAL EQUATIONS AND UNDERDETERMINED SYSTEMS

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Abstract. In this paper the linear algebraic system obtained from a singular integral equation with variable coefficients by a quadrature-collocation method is considered. We study this underdetermined system by means of the Moore Penrose generalized inverse. Convergence in compact subsets of $[-1,1]$ can be shown under some assumptions on the coefficients of the equation.

Key words. Singular integral equation, underdetermined system, generalized inverse

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1. Introduction.. @ < 11111111111 <<

The purpose of this study is the investigation of some properties of the underdetermined linear algebraic system obtained from singular integral equations (SIE's) by direct quadrature-collocation based on nonclassical nodes. We are able to characterize some properties of the system.

Contrary to what happens for Fredholm equations, in SIE's because of the presence of the Cauchy principal value singularity the n quadrature nodes used to discretize the singular integral cannot be employed to collocate the functional equation resulting from the previous operation. Other nodes are thus necessary. If this other set of nodes is chosen appropriately, it turns out that they need to be the $n-\kappa$ zeros of a certain orthogonal polynomial of a family related to the one that provides the quadrature nodes.

Recent investigations, see [15, 14, 8], have dealt with the problem of studying the effect of a "suboptimal" choice of the second set of nodes. In case of constant coefficients, we replace appropriate Gauss-Jacobi nodes with the more easily generated Chebyshev nodes [15, 14].

For variable coefficients, the quadrature and collocation nodes arise from nonclassical families of orthogonal polynomials, [4], and are difficult to construct. A recently proposed solution scheme replaces these nonclassical families by standard Gauss-Jacobi orthogonal polynomials, [8]. In both cases, the value of the unknown function at the collocation set is nonzero and hence cannot be ignored. Thus extra unknowns arise using this procedure. In order to obtain a square system, another formula needs to be used. It can be a quadrature of a different type, applied however using the same nodes, or an interpolatory formula relating the value of the unknown function at the two sets of nodes. These procedures have the drawback that they double the size of the discretized linear algebraic system.

If we want to use a "standard approach", there is the need of using nonclassical orthogonal polynomials, whose efficient calculation is not yet available. An alternative is to try to avoid the use of these polynomials by looking only at the asymptotics of the equation, i.e. at the singular endpoint behavior, expressed by the constants α and β defined in the next paragraph. The disadvantage however is the fact that for variable coefficient equations, the quantity $-(\alpha+\beta)$ is not an integer, the index of the equation, as it happens in the case of constant coefficients. We try here a different approach, by avoiding to look directly at the asymptotics, retaining however the use of classical Jacobi polynomials. Under suitable assumptions on the coefficients, we are able to show convergence in compact subsets of $[-1,1]$, but the price we pay consists in the convergence rate being affected by the use of this "unprecise" endpoint singularity information and by the growth of the error constant.

The note is organized as follows. In the next section we give the mathematical description of the problem. Section 3 is devoted to the presentation of the numerical method, and the last section contains the results of the analysis.

2. Preliminaries. We consider here the dominant singular integral equation with variable coefficients

$$a(x)\,\phi(x)+\frac{b(x)}{\pi}\int_{-1}^{1}\frac{\phi(t)}{t-x}dt=f(x)\qquad -1<x<1. \tag{1}$$

Let $H_\mu[-1,1]$ denote the class of Hölder continuous functions of exponent μ on $[-1,1]$, i.e. the functions $y(x)$ that satisfy the Hölder condition (H-condition)

$$|y(x)-y(t)|\le C^0\,|x-t|^\mu,$$

for a suitable constant $C^0>0$, and $0<\mu\le 1$. We assume that the coefficients are real valued functions on $H_\mu[-1,1]$, satisfying $r^2(x)=a^2(x)+b^2(x)>0$, for every $x\in[-1,1]$. By choosing a continuous path for the function

$$\log\frac{a(x)-ib(x)}{a(x)+ib(x)},$$

it is then possible to find integers M and N such that the quantities α_0 and β_0 given by

$$\alpha_0=\frac{1}{2\pi i}\log\frac{a(1)-ib(1)}{a(1)+ib(1)}+M,\ \ \beta_0=-\frac{1}{2\pi i}\log\frac{a(-1)-ib(-1)}{a(-1)+ib(-1)}+N,$$

satisfy $|\alpha_0|,|\beta_0|<1$, for more details see [4]. The index of the equation is then defined as $\chi=-(M+N)$. The fundamental function of the problem can be represented as

$$Z(x)=(1-x)^{\alpha_0}\,(1+x)^{\beta_0}\,\Omega(x),$$

where $\Omega(x)$ is a positive smooth function on $[-1,1]$. Classically, a new unknown function $\varphi(x)$ is defined by

$$\phi(x) = Z(x)\,\varphi(x).$$

The singular operator in (1) transforms any function $\phi(x)$ locally satisfying the H-condition, into a new function $\zeta(x)$ which also locally satisfies the H-condition; hence the right hand side $f(x)$ must also locally satisfy the H-condition in order that the solution satisfies the H-condition, [11]. On the other hand, the solution $\phi(x)$ locally satisfies the H-condition if $f(x)$ is assumed to locally satisfy the H-condition [11]. Since both $a(x),\ b(x) \in H_\mu[-1,1]$, then also $\varphi(x) \in H_\mu[-1,1]$. One more final assumption is made on the coefficient $b(x)$, namely that it is a polynomial. More general conditions leading to this situation are discussed in [5], [6] and [1].

In the standard approach, use is made of families of nonclassical orthogonal polynomials with respect to the weight function $Z(x)$. The related quadrature nodes and weights are difficult to compute. A major difference with respect to equations with constant coefficients consists in the fact that $-(\alpha_0+\beta_0) \neq \chi$.

Our basic approach here is still to try to exploit properties of classical Jacobi polynomials, at the expense of having to deal with less smooth unknown functions. We indeed rewrite the function $\phi(x)$ in terms of a suitable Jacobi weight $\rho(x)$, to be specified below, in order to express explicitly the singular behavior at the endpoints in a "classical" manner, and a new unknown function $y^*(x)$ such that

$$\phi(x) = \rho(x)y^*(x). \tag{2}$$

The introduction of the function $\rho(x)$ tries to capture the singular endpoint behavior of the solution, but not completely. Specifically, the exponents α and β are chosen according to the following rule. Since $(\alpha_0,\beta_0) \in (-1,1)\times(-1,1)$, we orthogonally project (α_0,β_0) onto the line $y=-x+\kappa$, $\kappa \in \{-1,0,1\}$, to find the point (α,β). The integer κ is chosen so that $\kappa=-1$, if $\alpha_0,\ \beta_0>0$; $\kappa=1$ if $\alpha_0,\ \beta_0<0$, and $\kappa=0$ otherwise. To be specific, we have

$$\alpha = \frac{1}{2}(-\kappa+\alpha_0-\beta_0),\ \beta = \frac{1}{2}(-\kappa+\beta_0-\alpha_0),\ \rho(x) = (1-x)^\alpha(1+x)^\beta.$$

Notice that with this choice,

$$|\alpha_0-\alpha| < \frac{1}{2},\ |\beta_0-\beta| < \frac{1}{2},\ \ \max(\alpha,\beta) \geq -\frac{1}{2}. \tag{3}$$

It follows that

$$y^*(x) = (1-x)^{\alpha_0-\alpha}(1+x)^{\beta_0-\beta}u(x), \tag{4}$$

and if u is smooth,

$$y^* \in H_\lambda[-1,1],\ \text{with}\ \lambda = \min(|\alpha_0-\alpha|, |\beta_0-\beta|). \tag{5}$$

Recall indeed the inequality $|x^{\nu} - y^{\nu}| \leq |x-y|^{\nu}$, $-1 < \nu < 1$, see [10], p. 57.

The integral in (1) can be discretized by a classical Gaussian quadrature with weight $\rho(x)$. Let $P_n^{(\alpha,\beta)}(x)$ denote the Jacobi polynomial of degree n relative to the weight function $\rho(x)$, and $P_{n-\kappa}^{(-\alpha,-\beta)}(x)$ the one of degree $n-\kappa$ relative to the reciprocal weight $\rho^{-1}(x)$. Let also t_i represent the zeros of $P_n^{(\alpha,\beta)}(x)$ and s_j those of $P_{n-\kappa}^{(-\alpha,-\beta)}(x)$. We will need the following

Lemma 1 . The Lebesgue constant for interpolation to y^* on the zeros of the Jacobi polynomial grows like

$$n^{max(\alpha,\beta)+\frac{1}{2}}, \quad \text{for} \quad \max(\alpha,\beta) > -\frac{1}{2}; \quad \log n, \quad \text{for} \quad \max(\alpha,\beta) = -\frac{1}{2}.$$

Proof. See Theorem 14.4 of [13]. By (5) and (3) its assumptions hold. ∎

If $\chi \leq 0$, the theory of singular integral equations states that the solution is unique, but for existence, in case $\chi < 0$, the right hand side has to satisfy χ orthogonality conditions. If $\chi > 0$ instead, the solution is not unique. To determine it uniquely, we need extra conditions, which we take as

$$\frac{1}{\pi}\int_{-1}^{1} P_k^{(\alpha,\beta)}(t)\,\phi(t)\,dt = K_k, \;\; k = 0, ..., \chi - 1. \tag{6}$$

Equation (1) can be rewritten as follows

$$a(x)\,\rho(x)\,y^*(x) + \frac{b(x)}{\pi}\int_{-1}^{1}\frac{\rho(t)\,y^*(t)}{t-x}dt = f(x)\,. \tag{7}$$

To evaluate the singular integral in (7), we will use Hunter's method [7]. Let

$$\psi_n^{(\alpha,\beta)}(z) = \int_{-1}^{1}\rho(t)\frac{P_n^{(\alpha,\beta)}(t)}{t-z}dt = 2(z-1)^{\alpha}(z+1)^{\beta}q_n^{(\alpha,\beta)}(z), \;\; z \notin [-1,1],$$

where $q_n^{(\alpha,\beta)}$ represents the so called Jacobi function of the second kind. We can define the values of the function $\psi_n^{(\alpha,\beta)}(x)$ on the interval $[-1,1]$ as follows

$$\psi_n^{(\alpha,\beta)}(x) \equiv \frac{1}{2}\left\{\psi_n^{(\alpha,\beta)}(x+i0) + \psi_n^{(\alpha,\beta)}(x-i0)\right\}.$$

It can be expressed explicitly by means of the hypergeometric function, [13], but in this case using (2.1) of [9] it reduces to

$$\frac{1}{\pi}\int_{-1}^{1}\frac{P_n^{(\alpha,\beta)}(t)\,\rho(t)}{t-x}dt = \cot(\pi\alpha)\,P_n^{(\alpha,\beta)}(x)\,\rho(x) - \frac{2^{-\kappa}}{\sin(\pi\alpha)}P_{n-\kappa}^{(-\alpha,-\beta)}(x)\,. \tag{8}$$

Then Hunter's method takes the form

$$Q_n^*(y^*, x) = \sum_{i=1}^{n}\frac{w_i y^*(t_i)}{t_i - x} + \frac{\psi_n^{(\alpha,\beta)}(x)y^*(x)}{P_n^{(\alpha,\beta)}(x)}. \tag{9}$$

The singular integral is thus replaced by the above quadrature

$$\int_{-1}^{1} \frac{\rho(t)y^*(t)}{t-x}dt = Q_n^*(y^*,x) + \epsilon_G(x), \text{ for } x \in (-1,1) \tag{10}$$

and ϵ_G represents the quadrature error, and the quadrature weights have the explicit expressions

$$\begin{aligned} w_i &= \int_{-1}^{1} \rho(t) \frac{P_n^{(\alpha,\beta)}(t)}{(t-t_i)P_n^{(\alpha,\beta)\prime}(t_i)} dt & (11) \\ &= 2^{\alpha+\beta} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)\Gamma(n+\alpha+\beta+1)} \frac{2n+\alpha+\beta}{P_n^{(\alpha,\beta)\prime}(t_i)P_{n-\kappa}^{(\alpha,\beta)}(t_i)}, \qquad i = 1,2,\dots,n & (12) \end{aligned}$$

3. Discretization . Recall that $s_j \neq t_i$, $j = 1, ..., n-\kappa$, are the zeros of $P_{n-\kappa}^{(-\alpha,-\beta)}(x)$; using (8) let us define for $j = 1, \dots, n-\kappa$,

$$d_j \equiv a(s_j)\rho(s_j) + \frac{b(s_j)\psi_n^{(\alpha,\beta)}(s_j)}{\pi P_n^{(\alpha,\beta)}(s_j)} \equiv \rho(s_j)\left[a(s_j) + b(s_j)\cot(\pi\alpha)\right] . \tag{13}$$

From this, (10) and (9), collocating at the node points , we have

$$d_j y^*(s_j) + \frac{b(s_j)}{\pi}\sum_{i=1}^{n} \frac{w_i y^*(t_i)}{t_i - s_j} + \epsilon_{G,j} = f(s_j), \tag{14}$$

In case of positive index, the normalization conditions can also be discretized using standard Gauss-Jacobi quadrature over the same nodes and with the same weights. Thus

$$\sum_{i=1}^{n} w_i P_k^{(\alpha,\beta)}(t_i)\, y^*(t_i) + \epsilon_{G,k}^0 = K_k, \; k = 0, ..., \chi - 1, \tag{15}$$

where $\epsilon_{G,k}^0$ are the components of the error $\underline{\epsilon}_G^0$ of standard Gauss-Jacobi quadrature.

Let us introduce the unknown vector

$$\underline{y} = [y(s_1), \dots, y(s_{n-\kappa}), y(t_1), \cdots, y(t_n)]^t$$

approximating the exact solution vector

$$\underline{y^*} = [y^*(s_1), \cdots, y^*(s_{n-\kappa}), y^*(t_1), \cdots, y^*(t_n)]^t .$$

In other words, we denote by $y^*(x)$ the exact solution of the original equation (7), together with (6), by $\underline{y^*}$ the vector of dimension $2n-\kappa$ of the function values of the exact solution y^* at the collocation points and the quadrature nodes. The vector $\underline{y}$ of dimension $2n-\kappa$ is the solution of the system (16) below.

After dropping the error term the discretized linear algebraic system of size $(n-\kappa+\chi)\times(2n-\kappa)$ can be written in block form as

$$\tilde{M}\underline{y} = \begin{bmatrix} D & A \end{bmatrix} \underline{y} = \underline{f} \tag{16}$$

where $\underline{f} = [f(s_1), \ldots, f(s_{n-\kappa}), K_0, ..., K_{\chi-1}]^t$. Here D is an $(n-\kappa+\chi)\times(n-\kappa)$ matrix with a special structure, where the only nonzero elements are

$$(D)_{j,j} = d_j, \quad \text{for } j = 1, \ldots, n-\kappa.$$

The matrix A has instead the following entries

$$A_{ij} = \begin{cases} \frac{b(s_i)w_j}{\pi(t_j - s_i)} & i = 1, ..., n-\kappa \\ w_j P_k^{(\alpha,\beta)}(t_j) & i = n+k+1-\kappa, \ k = 0, ..., \chi-1 \end{cases} \qquad j = 1, 2, \ldots, n.$$

From the original equation it follows

$$\tilde{M}\underline{y^*} + \underline{\epsilon}_G = \underline{f}, \tag{17}$$

where $\underline{\epsilon}_G$ denotes the consistency error vector, $\underline{\epsilon}_G = \left[\epsilon_{G,1}, ..., \epsilon_{G,n-\chi}, \epsilon^0_{G,0}, ..., \epsilon^0_{G,\chi-1}\right]^t$. Let I_n denote the $n\times n$ identity matrix, and let us define the square matrices of sizes n, and $2n-\kappa$ respectively

$$N = diag\left(w_j^{-1}\right), \ j = 1, ..., n; \ B = diag\left(I_{n-\kappa}, N^{\frac{1}{2}}\right).$$

Let us introduce

$$M = \tilde{M}B = \left[\begin{array}{cc} D & AN^{\frac{1}{2}} \end{array}\right]. \tag{18}$$

Rewrite the system (17)as follows

$$MB^{-1}\underline{y^*} + \underline{\epsilon}_G \equiv \underline{f} \tag{19}$$

and the system (16) as

$$MB^{-1}\underline{y} = \underline{f}. \tag{20}$$

This is a rectangular, underdetermined system. On taking its Moore-Penrose generalized inverse, we find a solution $\underline{y}$ satisfying (16) in the sense of minimizing $\left(B^{-1}\underline{y}\right)^t\left(B^{-1}\underline{y}\right)$

$$\underline{y} = BM^+\underline{f}. \tag{21}$$

We can also define $\underline{y}^*$ as a solution of (19) in the sense of minimizing $\left(B^{-1}\underline{y}^*\right)^t\left(B^{-1}\underline{y}^*\right)$

$$\underline{y}^* = BM^+\left(\underline{f} - \underline{\epsilon}_G\right). \tag{22}$$

Define the error as $\underline{e} = \underline{y}^* - \underline{y}$. Subtracting (21) from (22)

$$\underline{e} = -BM^+\underline{\epsilon}_G.$$

To determine convergence, we need estimates of the terms in the right hand side.

$$\|\underline{e}\| \le \|B\|\|M^+\|\|\underline{\epsilon}_G\|. \tag{23}$$

4. Estimates and Main Result. For the consistency error, and for some discussion of the norm of the Moore-Penrose generalized inverse, we need the results of [2]. Their quadrature however differs from (9). It is

$$Q_n\left(y^*,x\right)\equiv\sum_{i=1}^{n}W_i y^*\left(t_i\right)=\sum_{i=1}^{n}\frac{\psi_n^{(\alpha,\beta)}(t_i)-\psi_n^{(\alpha,\beta)}(x)}{P_n^{(\alpha,\beta)\prime}\left(t_i\right)\left(t_i-x\right)}y^*(t_i). \tag{24}$$

It can be easily recast in a form closely related to Hunter's. Upon collocation at s_j

$$Q_n\left(y^*,s_j\right)=\sum_{i=1}^{n}\frac{w_i y^*(t_i)}{t_i-s_j}-\cot\left(\pi\alpha\right)\rho\left(s_j\right)\sum_{i=1}^{n}\frac{P_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)\prime}\left(t_i\right)\left(t_i-s_j\right)}y^*\left(t_i\right), \tag{25}$$

where in the first sum the weights are given by (12). The relationship between weights is then

$$W_i=\frac{w_i}{t_i-s_j}-\cot\left(\pi\alpha\right)\rho\left(s_j\right)\frac{P_n^{(\alpha,\beta)}(s_j)}{P_n^{(\alpha,\beta)\prime}\left(t_i\right)\left(t_i-s_j\right)},$$

so that taking absolute values, summing and using lemma 1, for $\max\left(\alpha,\beta\right)>-\frac{1}{2}$,

$$\sum_{i=1}^{n}\left|\frac{w_i}{t_i-s_j}\right|\leq\sum_{i=1}^{n}\left|W_i\right|+\left|\cot\left(\pi\alpha\right)\rho\left(s_j\right)\right|n^{\max(\alpha,\beta)+\frac{1}{2}} \tag{26}$$

and similarly for $\max\left(\alpha,\beta\right)=-\frac{1}{2}$. Lemma 3 of [2] shows that the weights W_i in the quadrature behave like $H+K\log n$. However the constants H and K depend on the location of the singularity, i.e. in our situation, on the collocation points. We have the following three different cases. For the first weight, from (3.6) of [2]

$$\left|W_1\right|\leq\frac{B^*}{\left(1-s_j\right)\left(\frac{\alpha}{2}+\frac{3}{4}\right)\pi}\left\{1+O\left(n^{-1}\right)\right\},\ B^*=2^{|\alpha-\beta|+1}\left(1-s_j^2\right)^{-\frac{3}{4}}. \tag{27}$$

For the remaining ones, up to $j-1$, (3.7) gives

$$\sum_{k=1}^{j-2}\left|W_k\right|\leq\frac{B^*}{\left(1-s_j^2\right)\pi}\left\{D^*+\left(s_j+2\right)\log n\right\}\left\{1+O\left(n^{-1}\right)\right\}, \tag{28}$$

where the constant D^* is of the form

$$D^*=C_1+C_2\log\left(1-s_j^2\right).$$

In view of Theorem 8.9.1 of [13], we have

$$\log\left(1-s_j^2\right)\sim\alpha\log\left(j^{\alpha}n\right).$$

Substitution into (28) gives

$$\sum_{k=1}^{j-2} |W_k| \le \frac{B^*}{\left(1-s_j^2\right)\pi} \left\{C_1 + C_3 \alpha \log\left(j^\alpha n\right)\right\} \left\{1 + O\left(n^{-1}\right)\right\} \tag{29}$$

Finally for s_{j-1}, from (3.8) of [2]

$$|W_{j-1}| \le \frac{B^*}{\left(1-s_j^2\right)\pi} \left\{1 + O\left(n^{-1}\right)\right\}. \tag{30}$$

Similar results hold for the remaining nodes, by substituting α with β. On using again theorem 8.9.1 of [13] on the previous estimates (27), (29) and (30), together with (26), we have

Lemma 2. For the weights in Hunter's quadrature $Q_n^*\left(y^*, s_j\right)$ we have the bounds

$$\begin{aligned}
\left|\frac{w_1}{t_1 - s_j}\right| &\le C_4 \left(\frac{j}{n}\right)^{-2-\frac{3}{2}} \left\{1 + O\left(n^{-1}\right)\right\} \\
\sum_{i=1}^{j-2} \left|\frac{w_i}{t_i - s_j}\right| &\le C_5 \left(\frac{j}{n}\right)^{-2-\frac{3}{2}} \left\{1 + C_6 \log(n)\right\} \left\{1 + O\left(n^{-1}\right)\right\} \\
\left|\frac{w_{j-1}}{t_{j-1} - s_j}\right| &\le C_7 \left(\frac{j}{n}\right)^{-2-\frac{3}{2}} \left\{1 + O\left(n^{-1}\right)\right\}
\end{aligned}$$

and similary for the remaining weights.

Recall also formula (15.3.10) of [13] on the growth of the Christoffel numbers, i.e. the weights w_i relative to the nodes x_i, in ordinary Gauss-Jacobi quadrature

$$w_i \sim \frac{2^{\alpha+\beta+1}}{n}\pi \left(\sin\frac{\theta_i}{2}\right)^{2\alpha+1} \left(\cos\frac{\theta_i}{2}\right)^{2\beta+1} \sim \check{C} n^{-2(\alpha+\beta+1)},\ x_i = \cos\theta_i, \tag{31}$$

in view also of theorem 8.9.1 of [13]. It then follows

Lemma 3. For the matrix B, the following estimates hold

$$\|B\|_\infty = \|B\|_1 = \|B\|_2 \sim n^{1-\kappa}.$$

Proof. Use (31) and the definitions of κ and of the diagonal matrix B. ∎

Lemma 4. We have also,

$$\|M^t\|_\infty \le n\|M^t\|_1 \le n^{\frac{9}{2}+\min(\alpha,\beta,\mu)-\kappa} \log n.$$

Proof. In fact, $\|D_m^t\|_1 \le C_8 n^{\min(\alpha,\beta,\mu)}$, in view of the Hölder continuity of the function ρ and of a, b. For $N_m^{t/2}$ use (31) and for A_m^t use lemma 2, to get

$$\|A_m^t\|_1 \le n^{\frac{7}{2}} \log n.$$

Hence, the claim. □

Lemma 5. The matrix $\Lambda = ANA^t$ with entries given by (32), (33) and (34) below, has the following block structure, where D_1 and D_2 are diagonal matrices

$$\Lambda = \left[\begin{array}{cc} D_1 & G^t \\ G & D_2 \end{array} \right].$$

Proof. Let us calculate $\Lambda_{i,j}$. Let $\lambda_i = \pi \left[b\left(s_i\right)\right]^{-1}$. Since b is a polynomial with only finitely many zeros, for n large enough we can assume $b\left(s_i\right) \neq 0$. There are three different cases to consider.

For $1 \leq i \leq n-\kappa$, $n-\kappa+1 \leq j \leq n-\kappa+\chi$, letting $m = j-(n-\kappa+1) \in \{0, ..., \chi-1\}$, we have, by using the definition of the quadrature weigths (12)

$$\begin{aligned} \lambda_i \Lambda_{i,j} &= \sum_{k=1}^{n} \frac{1}{t_k - s_i} \frac{P_m^{(\alpha,\beta)}\left(t_k\right)}{P_n^{(\alpha,\beta)\prime}\left(t_k\right)} \frac{1}{\pi} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}\left(t\right) \rho\left(t\right)}{t - t_k} dt \\ &= \frac{1}{\pi} \sum_{k=1}^{n} \frac{P_m^{(\alpha,\beta)}\left(t_k\right)}{P_n^{(\alpha,\beta)\prime}\left(t_k\right)} \int_{-1}^{1} \left[\frac{1}{t-t_k} + \frac{1}{t_k - s_i}\right] \frac{P_n^{(\alpha,\beta)}\left(t\right) \rho\left(t\right)}{t - s_i} dt \end{aligned}$$

Observe that the interpolation formula is exact on polynomials of degree up to n, so that

$$\sum_{k=1}^{n} \frac{P_m^{(\alpha,\beta)}\left(t_k\right) P_n^{(\alpha,\beta)}\left(t\right)}{P_n^{(\alpha,\beta)\prime}\left(t_k\right)\left(t - t_k\right)} = P_m^{(\alpha,\beta)}\left(t\right), \quad \sum_{k=1}^{n} \frac{P_m^{(\alpha,\beta)}\left(t_k\right) P_n^{(\alpha,\beta)}\left(s_i\right)}{P_n^{(\alpha,\beta)\prime}\left(t_k\right)\left(s_i - t_k\right)} = P_m^{(\alpha,\beta)}\left(s_i\right).$$

Substituting into the former expressions we have

$$\lambda_i \Lambda_{i,j} = \frac{1}{\pi} \int_{-1}^{1} \frac{P_m^{(\alpha,\beta)}\left(t\right) \rho\left(t\right)}{\left(t - s_i\right)} dt - \frac{1}{\pi} \frac{P_m^{(\alpha,\beta)}\left(s_i\right)}{P_n^{(\alpha,\beta)}\left(s_i\right)} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}\left(t\right) \rho\left(t\right)}{t - s_i} dt;$$

by using the second kind Jacobi function (8) and in view of the choice of the collocation nodes, $P_{n-\kappa}^{(-\alpha,-\beta)}\left(s_i\right) = 0$, so that for $i \neq j$

$$\Lambda_{i,j} \equiv \left(G^t\right)_{i,m+1} \equiv \frac{2^{-\kappa} b\left(s_i\right)}{\sin\left(\pi\alpha\right)} P_{m-\kappa}^{(-\alpha,-\beta)}\left(s_i\right). \tag{32}$$

where we have introduced the $\chi \times (n-\kappa)$ matrix G.

For $i, j \leq n-k$, and $i \neq j$

$$\sum_{k=1}^{n} \frac{w_k}{t_k - s_i} \frac{1}{w_k} \frac{w_k}{t_k - s_j} = \frac{1}{s_i - s_j} \sum_{k=1}^{n} \left[\frac{w_k}{t_k - s_i} - \frac{w_k}{t_k - s_j}\right] = 0$$

in view of the previous computations, for the case $m \equiv 0$. For $i = j$ instead, we have, [8]

$$\Lambda_{i,i} = \frac{b^2\left(s_i\right)}{\pi} \sum_{k=1}^{n} \frac{w_k}{\left(t_k - s_i\right)^2} = \frac{b^2\left(s_i\right)}{w_i^*} \equiv a_i^2, \;\; i = 1, ..., n-\kappa. \tag{33}$$

We now consider the case $n-\kappa+1 \leq i,j \leq n-\kappa+\chi$; once again let $m = j-(n-\kappa)$, $q = i-(n-\kappa+1)$. Using the expression of the weights, the Lagrange interpolation formula being exact on polynomials of degree $m+q < n$, and the exactness of the quadrature over polynomials of degree "low enough",

$$\begin{aligned}
\Lambda_{i,j} &= \sum_{k=1}^{n} w_k P_q^{(\alpha,\beta)}(t_k) P_m^{(\alpha,\beta)}(t_k) \\
&= \frac{1}{\pi} \sum_{k=1}^{n} \frac{1}{P_n^{(\alpha,\beta)\prime}(t_k)} \int_{-1}^{1} \frac{P_n^{(\alpha,\beta)}(t)\,\rho(t)\,dt}{t-t_k} P_q^{(\alpha,\beta)}(t_k) P_m^{(\alpha,\beta)}(t_k) \\
&= \frac{1}{\pi} \int_{-1}^{1} \rho(t) \sum_{k=1}^{n} P_q^{(\alpha,\beta)}(t_k) P_m^{(\alpha,\beta)}(t_k) \frac{P_n^{(\alpha,\beta)}(t)}{P_n^{(\alpha,\beta)\prime}(t_k)(t-t_k)} dt \\
&= \frac{1}{\pi} \int_{-1}^{1} \rho(t) P_q^{(\alpha,\beta)}(t) P_m^{(\alpha,\beta)}(t)\,dt.
\end{aligned}$$

From the orthogonality of the Jacobi polynomials, then

$$\Lambda_{i,j} = \delta_{ij} \| P_q^{(\alpha,\beta)} \|_\rho^2 \equiv b_q^2, \quad q = 0, ..., \chi-1, \tag{34}$$

where the last symbol denotes the weighted two norm. ∎

Let us put $\bar{\Lambda} = DD^t + ANA^t$. We will need an estimate for $\bar{\Lambda}^{-1}$. Recall (7.32.2) of [13], for which

$$\| P_n^{(\alpha,\beta)} \|_\rho^2 = \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\,\Gamma(n+\beta+1)}{\Gamma(n+1)\,\Gamma(n+\alpha+\beta+1)} \tag{35}$$

so that

$$\| P_n^{(\alpha,\beta)} \|_\rho^{-2} \sim n. \tag{36}$$

Let us put $D_0 = diag(d_1, .., d_{n-\kappa})$, $E \equiv DD^t = diag\left(D_0^2, 0, ..., 0\right)$, a square matrix of size $n-\kappa+\chi$, as is $\bar{\Lambda}$. Then $\bar{\Lambda} = \Lambda + E$. Now Λ is a symmetric matrix, with the block form given in the lemma, where G is the $\chi \times (n-\kappa)$ matrix with elements $G_{m,i}$, given by (32) and the diagonal matrix are square, of sizes respectively $n-\kappa$ and χ, and whose elements are $D_1^2 = diag\left(a_1^2, ..., a_{n-\kappa}^2\right)$, see (33) and $D_2 = diag\left(b_0^2, ..., b_{\chi-1}^2\right)$, see (34). We will use a second decomposition of $\bar{\Lambda}$, $\bar{\Lambda} = \Lambda^* + \tilde{E}$

$$\Lambda^* = \begin{bmatrix} D_1^2 & G^t \\ G & D_2 \end{bmatrix}, \ \tilde{E} = \begin{bmatrix} D_0^2 & O \\ O & O \end{bmatrix}.$$

We need a preliminary result.

Lemma 6. For n large enough

$$\bar{\Delta} \equiv G D_1^{-2} G^t = \gamma \; diag\left(\| P_{-\kappa}^{(-\alpha,-\beta)} \|_{\rho^{-1}}^2, ..., \| P_{\chi-1-\kappa}^{(-\alpha,-\beta)} \|_{\rho^{-1}}^2 \right), \quad with \quad \gamma = \frac{2^{-2\kappa}}{\sin^2(\pi\alpha)}.$$

Proof. Observe that for $l, r = 0, ..., \chi - 1$

$$\begin{aligned} \bar{\Delta}_{l+1,r+1} &= \sum_{k=1}^{n-\kappa} \frac{2^{-\kappa}}{\sin(\pi\alpha)} P_{l-\kappa}^{(-\alpha,-\beta)}(s_k)\, w_k^* \frac{2^{-\kappa}}{\sin(\pi\alpha)} P_{r-\kappa-(n-\kappa)}^{(-\alpha,-\beta)}(s_k) \\ &= \gamma \sum_{k=1}^{n-\kappa} w_k^* P_{l-\kappa}^{(-\alpha,-\beta)}(s_k)\, P_{r-\kappa-(n-\kappa)}^{(-\alpha,-\beta)}(s_k). \end{aligned}$$

Using the definition of weights in the last expression we obtain

$$\begin{aligned} \bar{\Delta}_{l+1,r+1} &= \gamma \sum_{k=1}^{n-\kappa} \int_{-1}^{1} \rho^{-1}(t) \frac{P_{n-\kappa}^{(-\alpha,-\beta)}(t)\, dt}{P_{n-\kappa}^{(-\alpha,-\beta)\prime}(s_k)\,(t-s_k)} P_{l-\kappa}^{(-\alpha,-\beta)}(s_k)\, P_{r-\kappa-(n-\kappa)}^{(-\alpha,-\beta)}(s_k) \\ &= \gamma \int_{-1}^{1} \rho^{-1}(t) \sum_{k=1}^{n-\kappa} P_{l-\kappa}^{(-\alpha,-\beta)}(s_k)\, P_{r-\kappa-(n-\kappa)}^{(-\alpha,-\beta)}(s_k) \frac{P_{n-\kappa}^{(-\alpha,-\beta)}(t)}{P_{n-\kappa}^{(-\alpha,-\beta)\prime}(s_k)\,(t-s_k)} dt \\ &= \gamma \int_{-1}^{1} \rho^{-1}(t)\, P_{l-\kappa}^{(-\alpha,-\beta)}(t)\, P_{r-\kappa-(n-\kappa)}^{(-\alpha,-\beta)}(t)\, dt, \end{aligned}$$

the last step being exact if $n > l + r - \kappa$, since we interpolate a polynomial. It follows

$$\bar{\Delta}_{l+1,r+1} = \gamma \delta_{l,r} \|P_{l-\kappa}^{(-\alpha,-\beta)}\|_{\rho^{-1}}^2.$$

□

By introducing the elementary matrix

$$R = \begin{bmatrix} I & O \\ -GD_1^{-2} & I \end{bmatrix},$$

we can diagonalize Λ^*, since $\Lambda^* = RWR^t$, with $W = diag\,[D_1^2, D_2 - \bar{\Delta}]$. Thus the eigenvalues of Λ^* are just the entries on the diagonal of W. Observe that for $k = 0, ..., \chi - 1$, by (35)

$$\gamma \|P_{k-\kappa}^{(-\alpha,-\beta)}\|_{\rho^{-1}}^2 = \frac{1}{\sin^2(\pi\alpha)} \|P_k^{(\alpha,\beta)}\|_\rho^2 \tag{37}$$

so that the eigenvalues of Λ^* can be written down in an ordered fashion as follows

$$\gamma^* \|P_0^{(\alpha,\beta)}\|_\rho < ... < \gamma^* \|P_{\chi-1}^{(\alpha,\beta)}\|_\rho < 0 < \min_i a_i^2 < ... < \max_i a_i^2,\ \gamma^* = 1 - \frac{1}{\sin^2(\pi\alpha)}. \tag{38}$$

Since $\bar{\Lambda}$ is symmetric, so is its inverse. Their 2-norms are then given by their respective spectral radii. We are therefore interested in the eigenvalue of Λ^* which is closest to zero. In absence of more specific informations about the coefficients $a(x)$ and $b(x)$ of the original equation appearing in the terms a_i and d_i, the above task seems to be a difficult one. We will then make some simplifying assumptions in the following discussion.

Fix now $0 < \epsilon < 1$, and let $\Delta_\epsilon \equiv [-1+\epsilon, 1-\epsilon]$. First of all, we assume that no s_i tends to a zero of $b(x)$ as $n \longrightarrow \infty$. In view of (31) it then follows that $a_i^2 \sim b(s_i)\, n^{2(1-\kappa)} \sim n^{2-2\kappa}$. We can then say that as n increases all the a_i increases as well; hence it is reasonable to take the eigenvalue of Λ^* closest to zero to be $\omega_\chi = \gamma^* \|P_{\chi-1}^{(\alpha,\beta)}\|_\rho$; alternatively, this certainly holds if

$$|\omega_\chi| < \min_i a_i^2. \tag{39}$$

By the corollary to Weyl's theorem, see [12] p. 193, and the fact that $\tilde{E}$ is positive semidefinite, it follows that the eigenvalue of $\bar{\Lambda}$ closest to zero is bounded below by ω_χ and by a standard result, [12] p. 191, it is bounded above by $\omega_\chi + \|\tilde{E}\|_2$. We would like this quantity still to be negative, as this ensures that $\bar{\Lambda}$ is invertible and that this is still the eigenvalue of the matrix closest to zero. Hence together with (39) let us assume that

$$\omega_\chi + \|\tilde{E}\|_2 < 0. \tag{40}$$

To understand how strong this assumption could be, let us remark that $\|\tilde{E}\|_2 = \max d_i^2(s_i)$. Use then (13); since the coefficients are continuous, the quantity in (13) within the brackets is bounded above; however if α or β are negative the function ρ blows up like $n^{-4\min(\alpha,\beta)}$, when $s_i \longrightarrow \pm 1$. To ensure then that (40) holds, in this case we need then to restrict attention to $s_i \in \Delta_\epsilon$. However condition (40) may be satisfied even in $[-1,1]$ if both α, $\beta \geq 0$. In this case (40) does not appear to be a very strict requirement. We may even relax it a bit as follows

$$\left|\omega_\chi + \|\tilde{E}\|_2\right| \leq C_9 n^{-\zeta}, \quad > 0,$$

since more generally it involves the coefficients of the original equation. With this condition, in view of the above discussion, the following estimate holds

$$\|\bar{\Lambda}^{-1}\|_\infty \leq \sqrt{n}\|\bar{\Lambda}^{-1}\|_2 \leq C_{10} n^{\zeta+\frac{1}{2}}. \tag{41}$$

We have then

Lemma 7. For the consistency error the following estimates hold

$$\begin{aligned} \|\underline{\epsilon}_G\|_\infty &\leq C_{11} n^{\max(\alpha,\beta)+\frac{1}{2}+\lambda}, \quad \mathit{for}\ \max(\alpha,\beta) > -\frac{1}{2}, \\ \|\underline{\epsilon}_G\|_\infty &\leq C_{11} n^{\lambda} \log n, \quad \mathit{for}\ \max(\alpha,\beta) = -\frac{1}{2}. \end{aligned}$$

In Δ_ϵ, the above estimates can be written as

$$\begin{aligned} \|\underline{\epsilon}_G\|_{\epsilon,\infty} &\leq C_{11}(\epsilon)\, n^{\max(\alpha,\beta)+\frac{1}{2}-\eta}, \quad \mathit{for}\ \max(\alpha,\beta) > -\frac{1}{2}, \\ \|\underline{\epsilon}_G\|_{\epsilon,\infty} &\leq C_{11}(\epsilon)\, n^{-\eta} \log n, \quad \mathit{for}\ \max(\alpha,\beta) = -\frac{1}{2}, \end{aligned}$$

with η "arbitrarily" high, where the notation emphasizes the fact that the bound itself depends on ϵ, and the norm is taken in Δ_ϵ.

Proof. Standard techniques estimate the Gaussian quadrature error in terms of the Lebesgue constant times the best approximation error for the integrand, $E_n(y^*)$. Use lemma 1 and Jackson's theorem, which gives the best approximation error in terms of the modulus of continuity of the integrand. Recalling section 2, the integrand function $y^*(x)$ however is only in $H_\lambda[-1,1]$, see (5). This is not enough to ensure convergence for the method, as it will be clear in the proof of the theorem below. To obtain convergence, we are then forced once again to restrict the domain of interest to Δ_ϵ, since there $y^*(x)$ is analytic. We can then take $y^* \in C^\eta(\Delta_\epsilon)$, with η "arbitrarily" high. We thus have

$$E_{\epsilon,n}(y^*) \le C_{11}(\epsilon)\, n^{-\eta}.$$

□

Theorem 8. The proposed method is convergent in Δ_ϵ, with rate given by

$$\|\underline{e}\|_\infty \le C_{12}(\epsilon)\, n^{\frac{13}{2}+\zeta+\max(\alpha,\beta)+\min(\alpha,\beta,\mu)-2\kappa-\eta} \log n$$

Proof. We can recast the error equation (23) in the following form and use lemmas 3, 4 and 7 and (41) together with the equivalence of norms to get the claim

$$\|\underline{e}\|_\infty \le \|B\|_\infty \|M^t\|_\infty \|\bar{\Lambda}^{-1}\|_\infty \|\underline{\epsilon}_G\|_{\epsilon,\infty}.$$

□

Remarks.

1. In practice, at least for the nodes closest to the endpoints, the numerical convergence may very well be destroyed, since the error bound $C_{12}(\epsilon)$ does actually grow as ϵ tends to 0.
2. For nonpositive index equations the analysis simplifies considerably, since the matrix of the system is just given by (18). Then $\|\bar{\Lambda}^{-1}\|_\infty = \|D_3^{-1}\|_\infty = n^{\min(4\alpha,4\beta,\xi)}$, where $\xi > 0$ is the rate for which some s_i tend to a zero of $b(x)$, if they do it at all.
3. The result of the theorem shows that convergence can be attained only for the "central" nodes of $[-1,1]$; problems at the endpoints can in a certain sense be expected in view of our choice of the quadrature weight ρ, see the discussion after (2).

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