J. KSIAM **Vol.2, No.2, 57-65, 1998**

ON SIMPSON'S QUADRATURE FORMULA FOR DIFFERENTIABLE MAPPINGS WHOSE DERIVATIVES BELONG TO L $_p$− SPACES AND APPLICATIONS

Sever Silvestru Dragomir

Abstract

An estimation of remainder for Simpson's quadrature formula for differentiable mappings whose derivatives belong to L_p−spaces and applications in theory of special means (logarithmic mean, identric mean etc...) are given.

1 INTRODUCTION

The following inequality is well known in the literature as the *Simpson's inequality* :

$$\left| \int_a^b f(x)dx - \frac{b-a}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] \right| \leq \frac{1}{2880} \parallel f^{(4)} \parallel_\infty (b-a)^5 \quad (1.1)$$

where the mapping $f:[a,b]\to R$ **is supposed to be four time differentiable on the interval** (a,b) **and having the fourth derivative bounded on** (a,b)**, that is**

$$\parallel f^{(4)} \parallel_\infty := \sup_{x\in(a,b)} \mid f^{(4)}(x) \mid < \infty.$$

Now, if we assume that $I_h : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ **is a partition of the interval** $[a,b]$ **and** f **is as above, then we have the *Simpson's quadrature formula*:**

$$\int_a^b f(x)dx = A_S(f,I_h) + R_S(f,I_h) \quad (1.2)$$

where $A_S(f,I_h)$ **is the *Simpson's rule***

$$A_S(f,I_h) =: \frac{1}{6}\sum_{i=0}^{n-1}[f(x_i)+f(x_{i+1})]h_i + \frac{2}{3}\sum_{i=0}^{n-1} f\left(\frac{x_i+x_{i+1}}{2}\right)h_i \quad (1.3)$$

[0] AMS Subject Classification: 26D16, 26D99, 41A55.
Key Words and Phrases: Simpson's formula, Special Means.

and the *remainder term* $R_S(f, I_h)$ satisfies the estimation

$$| R_S(f, I_h) | \leq \frac{1}{2880} \parallel f^{(4)} \parallel_\infty \sum_{i=0}^{n-1} h_i^5 \tag{1.4}$$

where $h_i := x_{i+1} - x_i$ **for** $i = 0, ..., n-1$.
When we have an equidistant partitioning of $[a, b]$ **given by**

$$I_n : x_i := a + \frac{b-a}{n} i, i = 0, ..., n \tag{1.5}$$

then we have the formula

$$\int_a^b f(x)dx = A_{S,n}(f) + R_{S,n}(f) \tag{1.6}$$

where

$$A_{S,n}(f) := \frac{b-a}{6n} \sum_{i=0}^{n-1} [f(a + \frac{b-a}{n} i) + f(a + \frac{b-a}{n}(i+1))]$$

$$+ \frac{2(b-a)}{3n} \sum_{i=0}^{n-1} f(a + \frac{b-a}{n} \cdot \frac{2i+1}{2}) \qquad (1.7).$$

and the remainder satisfies the estimation

$$| R_{S,n}(f) | \leq \frac{1}{2880} \cdot \frac{(b-a)^5}{n^4} \parallel f^{(4)} \parallel_\infty . \tag{1.8}$$

In the recent paper [1] the author proved the following result for lipschitzian mappings

THEOREM 1.1. ***Let*** $f : [a, b] \to R$ ***be an*** $L-$***lipschitzian mapping on*** $[a, b]$. ***Then we have the inequality***

$$| \int_a^b f(x)dx - \frac{b-a}{3} [\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] | \leq \frac{5}{36} L(b-a)^2. \tag{2.1}$$

The following corollary is useful in practice:

COROLLARY 1.2. ***Suppose that*** $f : [a, b] \to R$ ***is a differentiable mapping whose derivative is bounded on*** (a, b) ***, i.e.,***

$$\|f'\|_\infty := \sup_{x \in (a,b)} | f'(x) | < \infty.$$

Then we have the inequality

$$| \int_a^b f(x)dx - \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] | \le \frac{5}{36}\|f'\|_\infty (b-a)^2. \quad (2.5)$$

In the paper [2], S.S. Dragomir proved a version of Simpson's inequality for mappings with bounded variation as follows:

THEOREM 1.3. ***Let*** $f : [a,b] \to R$ ***be a mapping with bounded variation on*** $[a,b]$. ***Then we have the inequality***

$$| \int_a^b f(x)dx - \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] | \le \frac{1}{3}(b-a)V_a^b(f) \quad (2.1)$$

where $V_a^b(f)$ ***denotes the total variation of*** f ***on the interval*** $[a,b]$.
The constant $\frac{1}{3}$ ***is the best possible one.***

The following corollary is useful in practice

COROLLARY 1.4. ***Suppose that*** $f : [a,b] \to R$ ***is a differentiable mapping whose derivative is integrable on*** (a,b) ***, i.e.,***

$$\|f'\|_1 := \int_a^b | f'(x) | dx < \infty.$$

Then we have the inequality

$$| \int_a^b f(x)dx - \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] | \le \frac{1}{3}\|f'\|_1 (b-a)^2. \quad (2.5)$$

For some other integral inequalities see the recent book [3].

The main aim of this paper is to point out some bounds of the remainder in terms of $p-$**norm of the derivative** f' **and apply them for composite quadrature formulae and for special means.**

2 SIMPSON'S INEQUALITY IN TERMS OF p-NORMS

The following result holds:

THEOREM 2.1. ***Let*** $f : [a, b] \to R$ ***be a differentiable mapping on*** (a, b) ***whose derivative belongs to*** $L_p(a, b)$. ***Then we have the inequality***

$$| \int_a^b f(x)dx - \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] |$$

$$\leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}}(b-a)^{1+\frac{1}{q}} \parallel f' \parallel_p \quad (2.1)$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1.$

Proof. **Using the integration by parts formula we have:**

$$\int_a^b s(x)f'(x)dx = \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] - \int_a^b f(x)dx \quad (2.2)$$

where

$$s(x) := \begin{cases} x - \frac{5a+b}{6}, x \in [a, \frac{a+b}{2}) \\ x - \frac{a+5b}{6}, x \in [\frac{a+b}{2}, b] \end{cases}.$$

Indeed,

$$\int_a^b s(x)f'(x)dx = \int_a^{\frac{a+b}{2}} (x - \frac{5a+b}{6})f'(x)dx + \int_{\frac{a+b}{2}}^b (x - \frac{a+5b}{6})f'(x)dx$$

$$= [(x - \frac{5a+b}{6})f(x)]_a^{\frac{a+b}{2}} + [(x - \frac{a+5b}{6})f(x)]_{\frac{a+b}{2}}^b - \int_a^b f(x)dx$$

$$= \frac{b-a}{3}[\frac{f(a)+f(b)}{2} + 2f(\frac{a+b}{2})] - \int_a^b f(x)dx$$

and the identity is proved.
Applying Hölder's integral inequality we get

$$| \int_a^b s(x)f'(x)dx | \leq (\int_a^b | s(x) |^q dx)^{\frac{1}{q}} \parallel f' \parallel_p . \quad (2.3)$$

Let us compute

$$\int_a^b \mid s(x) \mid^q dx = \int_a^{\frac{a+b}{2}} \mid x - \frac{5a+b}{6} \mid^q dx + \int_{\frac{a+b}{2}}^b \mid x - \frac{a+5b}{6} \mid^q dx$$

$$= \int_a^{\frac{5a+b}{6}} (\frac{5a+b}{6} - x)^q dx + \int_{\frac{5a+b}{6}}^{\frac{a+b}{2}} (x - \frac{5a+b}{6})^q dx$$

$$+ \int_{\frac{a+b}{2}}^{\frac{a+5b}{6}} (\frac{a+5b}{6} - x)^q dx + \int_{\frac{a+5b}{6}}^b (x - \frac{a+5b}{6})^q dx$$

$$= \frac{1}{q+1}[-(\frac{5a+b}{6} - x)^{q+1}|_a^{\frac{5a+b}{6}} + (x - \frac{5a+b}{6})^{q+1}|_{\frac{5a+b}{6}}^{\frac{a+b}{2}}$$

$$-(\frac{a+5b}{6} - x)^{q+1}|_{\frac{a+b}{2}}^{\frac{a+5b}{6}} + (x - \frac{a+5b}{6})^{q+1}|_{\frac{a+5b}{6}}^b]$$

$$= \frac{1}{q+1}[(\frac{5a+b}{6} - a)^{q+1} + (\frac{a+b}{2} - \frac{5a+b}{6})^{q+1}$$

$$+(\frac{a+5b}{6} - \frac{a+b}{2})^{q+1} + (b - \frac{a+5b}{6})^{q+1}]$$

$$= \frac{(2^{q+1}+1)(b-a)^{q+1}}{3(q+1)6^q}$$

Now, using the inequality (2.3) and the identity (2.2) we deduce the desired result (2.1). ■

The following corollary for Simpson's composite formula holds:

COROLLARY 2.2. ***Let*** f ***and*** I_h ***be as above. Then we have Simpson's rule (1.2) and the remainder*** $R_S(f, I_h)$ ***satisfies the estimation***

$$\mid R_S(f, I_h) \mid \leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} \parallel f' \parallel_p (\sum_{i=0}^{n-1} h_i^{1+q})^{\frac{1}{q}}. \quad (2.4)$$

Proof. **Apply Theorem 2.1 on the interval** $[x_i, x_{i+1}]$ $(i = 0, ..., n-1)$ **to get**

$$\left| \int_{x_i}^{x_{i+1}} f(x)dx - \frac{h_i}{3}[\frac{f(x_i)+f(x_{i+1})}{2} + 2f(\frac{x_i+x_{i+1}}{2})] \right|$$

$$\leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} \parallel f' \parallel_p h_i^{1+\frac{1}{q}} (\int_{x_i}^{x_{i+1}} \mid f'(t) \mid^p dt)^{\frac{1}{p}}.$$

Summing the above inequalities over i from 0 to $n-1$, using the generalized triangle inequality and Hölder's discrete inequality, we get

$$\mid R_S(f, I_h) \mid \leq \sum_{i=0}^{n-1} \left| \int_{x_i}^{x_{i+1}} f(x)dx - \frac{h_i}{3}[\frac{f(x_i)+f(x_{i+1})}{2} + 2f(\frac{x_i+x_{i+1}}{2})] \right|$$

$$\leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} \sum_{i=0}^{n-1} h_i^{1+\frac{1}{q}} (\int_{x_i}^{x_{i+1}} \mid f'(t) \mid^p dt)^{\frac{1}{p}}$$

$$\leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} (\sum_{i=0}^{n-1} (h_i^{1+\frac{1}{q}})^q)^{\frac{1}{q}} \times (\sum_{i=0}^{n-1} ((\int_{x_i}^{x_{i+1}} \mid f'(t) \mid^p dt)^{\frac{1}{p}})^p)^{\frac{1}{p}}$$

$$= \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} \parallel f' \parallel_p (\sum_{i=0}^{n-1} h_i^{1+q})^{\frac{1}{q}}$$

and the corollary is proved. ■

The case of equidistant partitioning is embodied in the following corollary:

COROLLARY 2.4. ***Let f be as above and if I_n is an equidistant partitioning of $[a,b]$, then we have th estimation***

$$\mid R_{S,n}(f) \mid \leq \frac{1}{6n}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \parallel f' \parallel_p .$$

Remark 2.5. If we want to approximate the integral $\int_a^b f(x)dx$ by Simpson's formula $A_{S,n}(f)$ with an accuracy less that $\varepsilon > 0$, we need at least $n_\varepsilon \in N$ points for the division I_n, where

$$n_\varepsilon := [\frac{1}{6\varepsilon}(\frac{2^{q+1}+1}{3(q+1)})^{\frac{1}{q}} (b-a)^{1+\frac{1}{q}} \parallel f' \parallel_p] + 1$$

and $[r]$ denotes the integer part of $r \in R$.

Comments 2.6. If the mapping $f : [a,b] \to R$ is neither four time differentiable nor the fourth derivative is bounded on (a,b), then we can not apply the classical estimation in Simpson's formula using the fourth derivative. But if we assume that $f' \in L_p(a,b)$, then we can use instead the formula (2.4).

We give here a class of mappings whose first derivatives belong to $L_p(a,b)$ but having the fourth derivatives unbounded on the given interval.

Let $f_s : [a,b] \to R,\ f_s(x) := (x-a)^s$ where $s \in (3,4)$. Then obviously

$$f_s'(x) := s(x-a)^{s-1}, x \in (a,b)$$

and

$$f_s^{(4)}(x) = \frac{s(s-1)(s-2)(s-3)}{(x-a)^{4-s}}, x \in (a,b).$$

It is clear that $\lim_{x \to a+} f_s^{(4)}(x) = +\infty$ but $\|f_s'\|_p = s \frac{(b-a)^{s-1+\frac{1}{p}}}{((s-1)p+1)^{\frac{1}{p}}} < \infty$.

3 APPLICATIONS FOR SPECIAL MEANS

Let us recall the following means:

1. *Arithmetic mean*

$$A = A(a,b) := \frac{a+b}{2}, a, b \geq 0;$$

2. *Geometric mean*

$$G = G(a,b) := \sqrt{ab}, a, b \geq 0;$$

3. *Harmonic mean*

$$H = H(a,b) := \frac{2}{\frac{1}{a} + \frac{1}{b}}, a, b > 0;$$

4. *Logarithmic mean*

$$L = L(a,b) := \frac{b-a}{\ln b - \ln a}, a, b > 0, a \neq b;$$

5. ***Identric mean***

$$I = I(a,b) := \frac{1}{e}(\frac{b^b}{a^a})^{\frac{1}{b-a}}, a, b > 0, a \neq b;$$

6. ***p-Logarithmic mean***

$$S_p = S_p(a,b) := \left[\frac{b^{p+1} - a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}}, p \in R \backslash \{-1, 0\}, a, b > 0, a \neq b.$$

It is well known that S_p is monotonous nondecreasing over $p \in R$ with $S_{-1} := L$ and $S_0 := I$. In particular, we have the following inequalities

$$H \leq G \leq L \leq I \leq A. \quad (3.1)$$

In what follows, by the use of Theorem 2.1, we point out some new inequalities for the above means.

1. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = x^s, s \in R \backslash \{-1, 0\}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = S_s^s(a,b), f(\frac{a+b}{2}) = A^s(a,b), \frac{f(a)+f(b)}{2} = A(a^s, b^s)$$

and $\|f'\|_p = \mid s \mid S_{(s-1)p}^{s-1}(b-a)^{\frac{1}{p}}$.

Using the inequality (2.1) we get

$$\mid S_s^s(a,b) - \frac{1}{3}A(a^s,b^s) - \frac{2}{3}A^s(a,b) \mid \leq \frac{1}{6}[\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} \mid s \mid S_{(s-1)p}^{s-1}(a,b)(b-a). (3.2)$$

where $\frac{1}{p} + \frac{1}{q} = 1, p > 1$.

2. Let $f : [a,b] \to R$ $(0 < a < b), f(x) = \frac{1}{x}$. Then

$$\frac{1}{b-a}\int_a^b f(x)dx = L^{-1}(a,b), f(\frac{a+b}{2}) = A^{-1}(a,b),$$

$$\frac{f(a)+f(b)}{2} = H^{-1}(a,b) \text{ and } \|f'\|_p = S_{-2p}^{-2}(a,b)(b-a)^{\frac{1}{p}}$$

Using the inequality (2.1) we get

$$| 3HA - LA - 2LH | \leq \frac{1}{2} AHL [\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} S_{-2p}^{-2} (b-a) \qquad (3.3).$$

3. Let $f : [a, b] \to R$ $(0 < a < b), f(x) = \ln x$. **Then**

$$\frac{1}{b-a} \int_a^b f(x)dx = \ln I(a,b), f(\frac{a+b}{2}) = \ln A(a,b),$$

$$\frac{f(a)+f(b)}{2} = \ln A(a,b) \text{ and } \|f'\|_p = S_{-p}^{-1}(a,b)(b-a)^{\frac{1}{p}}.$$

.

Using the inequality (2.1) we get

$$| \ln [\frac{I}{G^{1/3}A^{2/3}}] | \leq \frac{1}{6} [\frac{2^{q+1}+1}{3(q+1)}]^{\frac{1}{q}} S_{-p}^{-1}(a,b)(b-a). \qquad (3.4)$$

REFERENCES

[1] **DRAGOMIR S.S.: On Simpson's quadrature formula for lipschitzian mappings and applications, *submitted.***

[2] **DRAGOMIR, S.S.: On Simpson's quadrature formula for mappings with bounded variation and applications, *submitted.***

[3] **MITRINOVIĆ, D.S.; PEČARIĆ, J.E. and FINK, A.M.: *Inequalities for Functions and Their Integrals and Derivatives,* Kluwer Academic Publishers, 1994.**