

# A Modified Adams Predictor-Corrector Method for Differential Equations with Highly Oscillating Solutions

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## Abstract

An algorithm for a solution of ordinary differential equations using a modified corrector in the Adams predictor-corrector method of order four is described. The Lagrange interpolation used in the corrector of the Adams method is replaced partially by the cubic spline interpolation satisfying the first derivative constraints at the two end points. By exhibiting three examples, we show that the proposed method is more efficient when the solution of a differential equation is highly oscillating.

## 1. Introduction

There are many studies on variations of the Adams predictor-corrector method. Some considers the accuracy measures[1] while most others consider the stability problems[2,3,4]. In this paper, we consider a variation of the method which is more efficient for a specific type of problems, i.e. the cases where the solutions are highly oscillating. Given a differential equation  $y' = f(x, y)$  with  $y_0 = y(x_0)$ , we may write the solution as

$$y_{n+1} = y_n + \int_{x_n}^{x_{n+1}} f(t, y(t))dt \quad (1)$$

In evaluating the integral, the Adams-Bashforth formula[5] of order  $k$  at  $x_n$  uses a polynomial  $p_{k,n}$  interpolating the computed derivatives at the  $k$  preceding points, i.e.

$p_{k,n}(x) = \sum_{j=1}^k l_j(x) f_{n+1-j}$  for  $j = 1, 2, \dots, k$ , where  $l_j(x)$ 's are the Lagrange polynomials defined on  $x_{n+1-j}$ ,  $j = 1, 2, \dots, k$ .

In the following, we consider the case of  $k = 4$  and assume that the points  $x_n$ 's are equally spaced with  $h = x_n - x_{n-1}$ . The Adams predictor-corrector method of order four used in this paper is of the form[5,p88]

$$(g_1, g_2, g_3, g_4, g_5) = \left(1, \frac{1}{2}, \frac{5}{12}, \frac{3}{8}, \frac{251}{720}\right)$$

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$$(\phi_1^*, \phi_2^*, \phi_3^*, \phi_4^*) = (f_n, f_n - f_{n-1}, f_n - 2f_{n-1} + f_{n-2}, f_n - 3f_{n-1} + 3f_{n-2} - f_{n-3})$$

$$\begin{aligned} p_{n+1} &= y_n + h \sum_{i=1}^4 g_i \phi_i^* \\ \phi_5^e &= 0, \quad \phi_i^e = \phi_{i+1}^e + \phi_i^*, \quad i = 4, 3, 2, 1 \\ f_{n+1}^p &= f(x_{n+1}, p_{n+1}) \\ y_{n+1} &= p_{n+1} + hg_5(f_{n+1}^p - \phi_1^e) \\ f_{n+1} &= f(x_{n+1}, y_{n+1}) \end{aligned} \quad (2)$$

Our variation is to add one more step at the end of (2) to redefine  $y_{n+1}$  by  $y_n + \int_{x_n}^{x_{n+1}} S_n(t) dt$ , where  $S_n(x)$  is the cubic spline interpolation which satisfies

$$\begin{aligned} S_n'(x_{n-2}) &= f'(x_{n-2}, y_{n-2}) = f_x(x_{n-2}, y_{n-2}) + y'(x_{n-2})f_y(x_{n-2}, y_{n-2}) \\ S_n(x_{n+2-j}) &= f_{n+2-j}, \quad j = 1, 2, 3, 4 \\ S_n'(x_{n+1}) &= f'(x_{n+1}, y_{n+1}) = f_x(x_{n+1}, y_{n+1}) + y'(x_{n+1})f_y(x_{n+1}, y_{n+1}) \end{aligned} \quad (3)$$

Note that the cubic spline function  $S_n(x)$  interpolates not only the function  $f(t, y(t))$  at the four points  $t_{n-2}, t_{n-1}, t_n, t_{n+1}$  but also its derivatives at  $t_{n-2}, t_{n+1}$ .

For convenience, we let  $a = x_n$ . Then the spline interpolation function  $S_n(x)$  can be written as

$$S_n(x) = \sum_{j=1}^6 c_j B_{a-(4-j)h}(x) \quad (4)$$

where  $B_{a-(4-j)h}(x)$ 's are the B-spline functions[6] with support  $[a-(6-j)h, a-(2-j)h]$ . Using properties of B-splines, the five equations in (2) can be written in a matrix form as  $A\mathbf{c} = \mathbf{b}$ , where  $\mathbf{c} = (c_1, c_2, \dots, c_6)^T$ ,  $\mathbf{b} = (f'_{n-2}, f_{n-2}, f_{n-1}, f_n, f_{n+1}, f'_{n+1})^T$ , and  $A$  is the coefficient matrix

$$A = \frac{1}{6} \begin{bmatrix} -\frac{3}{h} & 0 & \frac{3}{h} & 0 & 0 & 0 \\ 1 & 4 & 1 & 0 & 0 & 0 \\ 0 & 1 & 4 & 1 & 0 & 0 \\ 0 & 0 & 1 & 4 & 1 & 0 \\ 0 & 0 & 0 & 1 & 4 & 1 \\ 0 & 0 & 0 & -\frac{3}{h} & 0 & \frac{3}{h} \end{bmatrix} \quad (5)$$

When the integral of  $B_{a-(4-j)h}(x)$  over the interval  $[x_n, x_{n+1}] = [a, a+h]$  is evaluated, we find that for  $j = 3, 4, 5$ ,

$$\int_a^{a+h} B_{a-h}(t) dt = \int_a^{a+h} B_{a+2h}(t) dt = \frac{h}{24}, \quad \int_a^{a+h} B_a(t) dt = \int_a^{a+h} B_{a+h}(t) dt = \frac{11h}{24}$$

while the integrals of  $B_{a-3h}(x)$  and  $B_{a-2h}(x)$  are both zero. Therefore, the integral of the cubic spline interpolation function  $S_n(x)$  can be written as

$$\int_a^{a+h} S_n(t)dt = \int_a^{a+h} \sum_{j=1}^6 c_j B_{a-(4-i)h}(t)dt = \frac{h}{24}(c_3 + 11c_4 + 11c_5 + c_6) \quad (6)$$

## 2. A Modified Algorithm

The coefficients  $c_j$ 's of the spline interpolation function  $S_n(x)$  in (6) can be computed by using the inverse of the matrix  $A$  in (5) along with the relation  $\mathbf{c} = A^{-1}\mathbf{b}$ . When the inverse of  $A$  is computed, we have

$$A^{-1} = \frac{1}{45} \begin{bmatrix} -97h & -21 & 84 & -24 & 6 & -2h \\ 26h & 78 & -42 & 12 & -3 & h \\ -7h & -21 & 84 & -24 & 6 & -2h \\ 2h & 6 & -24 & 84 & -21 & 7h \\ -h & -3 & 12 & -42 & 78 & -26h \\ 2h & 6 & -24 & 84 & -21 & 97h \end{bmatrix} \quad (7)$$

and hence the coefficients  $c_j$ 's of the B-splines become

$$\begin{aligned} c_3 &= \frac{1}{45}(-7hf'_{n-2} - 21f_{n-2} + 84f_{n-1} - 24f_n + 6f_{n+1} - 2hf'_{n+1}) \\ c_4 &= \frac{1}{45}(2hf'_{n-2} + 6f_{n-2} - 24f_{n-1} + 84f_n - 21f_{n+1} + 7hf'_{n+1}) \\ c_5 &= \frac{1}{45}(-hf'_{n-2} - 3f_{n-2} + 12f_{n-1} - 42f_n + 78f_{n+1} - 26hf'_{n+1}) \\ c_6 &= \frac{1}{45}(2hf'_{n-2} + 6f_{n-2} - 24f_{n-1} + 84f_n - 21f_{n+1} + 97hf'_{n+1}) \end{aligned}$$

Substituting these into (6), we obtain

$$\int_a^{a+h} S_n(x)dx = \frac{h}{1080}(6hf'_{n-2} + 18f_{n-2} - 72f_{n-1} + 522f_n + 612f_{n+1} - 114hf'_{n+1}) \quad (8)$$

## 3. Examples

The following are some of the computed results using the relation (6) as a corrector for the Adams-Moulton's method. The results show that when the solutions are highly oscillating, the modified corrector reduces the error substantially even though it does not for other cases.

**Example 1.**  $y' = y + 10e^x \text{Cos}(10x)$ ,  $y(0) = 0$ .

Note that the analytical solution of the above is  $y = e^x \text{Sin}(10x)$ . We solved the above equation from  $x = 0$  to  $x = 10$  using both the standard Adams-Moulton's

method and the modified method. Table 1 shows the maximum and average of the absolute errors for various values of  $h$ . For  $h \geq 0.05$ , both the maximum and average errors are reduced to less than one tenth of those from the standard method. In the case of  $h < 0.05$ , the differences become significant when the integration range gets larger than  $[0,10]$ .

**Table 1.** Comparison of Maximum and Average Errors - Example 1

| $h$            | 0.2          |             | 0.1          |             | 0.05         |             | 0.025        |             |
|----------------|--------------|-------------|--------------|-------------|--------------|-------------|--------------|-------------|
|                | <i>Adams</i> | <i>Mod.</i> | <i>Adams</i> | <i>Mod.</i> | <i>Adams</i> | <i>Mod.</i> | <i>Adams</i> | <i>Mod.</i> |
| <i>Average</i> | 929.4        | 65.1        | 27.52        | 2.54        | 2.15         | 0.83        | 0.40         | 0.33        |
| <i>Max.</i>    | 10224.4      | 1025.5      | 311.89       | 39.63       | 28.10        | 7.18        | 3.91         | 3.31        |

**Example 2.**  $y' = \frac{y}{x} + 2x^2 \text{Cos}(x^2)$ ,  $y(0) = 0$ ,  $y'(0) = 0$

It is easy to check that the analytical solution for the above equation is  $y = x \text{Sin}(x^2)$ . When the above equation is solved from  $x = 0$  to  $x = 10., 20., 30.$  respectively by using both the standard Adams-Moulton's method and the modified method, we obtain the results shown in Table 2.

**Table 2.** Comparison of Maximum and Average Errors - Example 2

|             | <i>Range</i> | $x = 0$ to 10 |             | $x = 0$ to 20 |             | $x = 0$ to 30 |             |
|-------------|--------------|---------------|-------------|---------------|-------------|---------------|-------------|
|             |              | <i>Avg.</i>   | <i>Max.</i> | <i>Avg.</i>   | <i>Max.</i> | <i>Avg.</i>   | <i>Max.</i> |
| $h = 0.1$   | <i>Adams</i> | 0.4106        | 3.731       | 9.123         | 65.894      | 46.107        | 597.2       |
|             | <i>Mod.</i>  | 0.0426        | 0.348       | 2.845         | 24.898      | 37.392        | 568.6       |
| $h = 0.025$ | <i>Adams</i> | .00066        | .00614      | .03690        | 0.3286      | 0.3243        | 3.298       |
|             | <i>Mod.</i>  | .00031        | .00135      | .00353        | 0.0345      | 0.0323        | .3165       |

**Example 3.**  $y' = 2x \text{Cos}(x^2)$ ,  $y(0) = 0$ .

It is clear that the analytical solution of the above is  $y = \text{Sin}(x^2)$ . The equation is solved from  $x = 0$  to  $x = 10., 20., 30.$  respectively by using both the standard Adams-Moulton's method and the modified method to obtain the results shown in Table 3.

**Table 3.** Comparison of Maximum and Average Errors - Example 3

|             | <i>Range</i> | $x = 0$ to 10 |             | $x = 0$ to 20 |             | $x = 0$ to 30 |             |
|-------------|--------------|---------------|-------------|---------------|-------------|---------------|-------------|
|             |              | <i>Avg.</i>   | <i>Max.</i> | <i>Avg.</i>   | <i>Max.</i> | <i>Avg.</i>   | <i>Max.</i> |
| $h = 0.1$   | <i>Adams</i> | .04777        | 0.3711      | 0.5607        | 3.2963      | 1.8798        | 19.91       |
|             | <i>Mod.</i>  | .00509        | 0.0373      | 0.1670        | 1.2636      | 1.4151        | 19.07       |
| $h = 0.025$ | <i>Adams</i> | .00006        | .00056      | 0.0019        | 0.0164      | 0.0127        | 0.1103      |
|             | <i>Mod.</i>  | .00001        | .00009      | 0.0002        | 0.0017      | 0.0013        | 0.0106      |

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