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CHEBYSHEV PSEUDOSPECTRAL-FINITE ELEMENT METHOD FOR TWO-DIMENSIONAL UNSTEADY NAVIER-STOKES EQUATION

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Abstract A Chebyshev pseudospectral-finite element method is proposed for two-dimensional unsteady Navier-Stokes equation. The generalized stability and the convergence are proved strictly. The numerical results show the advantages of this method.
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1 Introduction.

Spectral and pseudospectral methods have the high accuracy. In particular, pseudospectral method is easier to be performed. But in most practical problems, the domains are not rectangular. This fact limits their applications. However, the sections of domains might be rectangular in certain directions, such as a cylindrical container. For solving such problems, it is natural to use Chebyshev pseudospectral-finite element approximation, see [1]. In this paper, we develop a mixed Chebyshev pseudospectral-finite element method for two-dimensional unsteady Navier-Stokes equation. It is easy to generalize this method to three-dimensional problems with complex geometry. In particular, it is easy to deal with the nonlinear terms. We also follow the idea in [2, 3] to calculate the pressure based on a Poisson equation. Therfore we avoid the difficult job of choosing the trial function space in which the divergence of every element vanishes. We construct the scheme and present the numerical results in Section 2 and 3. The numerical results show the advantages of this method. We list some lemmas in Section 4, and then prove the generalized stability and the convergence in the last two sections.

2 The Scheme.

Let $I_x = \{x \;/\; -1 < x < 1\}$, $I_y = \{y \;/\; 0 < y < 1\}$ and $\Omega = I_x \times I_y$. The speed and the pressure are denoted by $U = (U_1, U_2)$ and P, respectively. $\nu > 0$ is the kinetic viscosity.

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U_0 and f are given functions. Let $T > 0$ and $\partial_z = \frac{\partial}{\partial z}$, $z = t, x, y$. We consider the problem

$$\begin{cases} \partial_t U + d(U,U) + \nabla P - \nu \nabla^2 U = f, & \text{in } \Omega \times (0,T], \\ \nabla^2 P + \Phi(U) = \nabla \cdot f, & \text{in } \Omega \times (0,T], \\ U|_{t=0} = U_0, & \text{in } \Omega \bigcup \partial\Omega \end{cases} \tag{2.1}$$

where

$$d(U,V) = \partial_x (V_1 U) + \partial_y (V_2 U), \quad \Phi(U) = 2(\partial_y U_1 \partial_x U_2 - \partial_x U_1 \partial_y U_2).$$

(2.1) is one of representations of Navier-Stokes equation. Suppose that the boundary is a non-slip wall and so $U = 0$ on $\partial\Omega$. There is no boundary condition for the pressure. But if we use the second formula of (2.1) to evaluate the pressure, we need a non-standard boundary condition. Sometimes, it is assumed approximately that $\frac{\partial P}{\partial n} = g(x)$ on $\partial\Omega$, see [2, 3]. For simplicity, let $g(x) = 0$ in the following discussions. In addition, for fixing the value of pressure, we require that for all $t \le T$,

$$\int\int_\Omega P(x,y,t)dxdy = 0.$$

Since the derivation of (2.1) implies the incompressibility, we avoid the difficult job of constructing the trial function space in which the divergence of every element vanishes.

Let D be an interval (or a domain) in $\mathcal{R}$ (or $\mathcal{R}^2$). $L^2(D)$, $H^r(D)$ and $H_0^r(D)$ $(r > 0)$ denote the usual Hilbert spaces with the usual inner products and norms. We also define

$$L_0^2(D) = \{u \in L^2(D) \ / \ \int_D u dD = 0\}.$$

Let $\omega(x) = (1 - x^2)^{-\frac{1}{2}}$ and

$$(u,v)_{\omega,I_x} = \int_{I_x} uv\omega dx, \qquad \|u\|^2_{\omega,I_x} = (u,u)_{\omega,I_x},$$

$$L^2_\omega(I_x) = \{u(x) \ / \ u \ \text{ is measurable on } I_x \text{ and } \|u\|_{\omega,I_x} < \infty\}.$$

Furthermore

$$(u,v)_\omega = \int\int_\Omega uv\omega dxdy, \quad \|u\|^2_\omega = (u,u)_\omega,$$

$$L^2_\omega(\Omega) = \{u(x,y) \ / \ u \text{ is measurable on } \Omega \text{ and } \|u\|_\omega < \infty\}.$$

Let $a_\omega(u,v) = (\nabla u, \nabla(v\omega))_{L^2(\Omega)}$. The weak formulation of (2.1) is to find $U \in X^2(\Omega)$ and $P \in Y(\Omega)$ for all $t \le T$ such that

$$\begin{cases} (\partial_t U + d(U,U) + \nabla P, v)_\omega + a_\omega(U,v) = (f,v)_\omega, & \forall v \in X^2(\Omega), \\ a_\omega(P,w) = (\Phi(U) - \nabla \cdot f, w)_\omega, & \forall w \in \tilde{X}(\Omega) \end{cases} \tag{2.2}$$

where

$$\tilde{X}(\Omega) = \{u \ / \ u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \in L^2_\omega(\Omega)\}, \qquad X(\Omega) = \tilde{X}(\Omega) \bigcap \{u \ / \ u = 0 \text{ on } \partial\Omega\},$$

$$Y(\Omega) = \tilde{X}(\Omega) \bigcap L^2_0(\Omega) \bigcap \{u \ / \ \frac{\partial u}{\partial x} = 0 \text{ for } |x| = 1\}.$$

We now construct the trial function spaces. For the Chebyshev approximation, let N be any positive integer and $\mathcal{P}_N(I_x)$ denote the set of all algebraic polynomials of degree $\leq N$, defined on I_x. Moreover

$$V_N(I_x) = \{u(x) \in \mathcal{P}_N(I_x) \ / \ u(-1) = u(1) = 0\},$$

$$W_N(I_x) = \{u(x) \in \mathcal{P}_N(I_x) \ / \ \frac{du}{dx}(-1) = \frac{du}{dx}(1) = 0\}.$$

For the finite element approximation, let τ_h be a class of regular decomposition of I_y with subintervals $I_l = (y_{l-1}, y_l)$, $1 \leq l \leq M$ where $0 = y_0 < y_1 < \cdots < y_M = 1$. Suppose that τ_h satisfies the inverse assumption. Let $h = \max_{1\leq l\leq M} |y_l - y_{l-1}|$, $\bar{h} = \min_{1\leq l\leq M} |y_l - y_{l-1}|$. Then $h/\bar{h} \leq d$, d being a positive constant independent of h. Moreover let $k \geq 1$ and

$$\tilde{S}^k_h(I_y) = \{v(y) \ / \ v|_{I_l} \in \mathcal{P}_k(I_l), 1 \leq l \leq M\}, \qquad S^k_h(I_y) = \tilde{S}^k_h(I_y) \bigcap H^1_0(I_y).$$

Then we take the spaces $X^k_{N,h}(\Omega)$ and $Y^k_{N,h}(\Omega)$ as the trial function spaces for the speed and the pressure respectively, defined as

$$X^k_{N,h}(\Omega) = V_N(I_x) \otimes S^k_h(I_y), \qquad Y^k_{N,h}(\Omega) = [W_N(I_x) \otimes (\tilde{S}^k_h(I_y) \bigcap H^1(I_y))] \bigcap L^2_0(\Omega).$$

In addition,

$$\tilde{X}^k_{N,h}(\Omega) = V_N(I_x) \otimes (\tilde{S}^k_h(I_y) \bigcap H^1(I_y)).$$

Next, let $x^{(j)}$ and $\omega^{(j)}$ be the nodes and weights of Gauss-Lobatto integration. The corresponding discrete inner products and norms are defined as

$$\begin{aligned}(u, v)_{N,\omega} &= \sum_{j=0}^{N} u(x^{(j)}) v(x^{(j)}) \omega^{(j)}, \qquad \|u\|^2_{N,\omega} = (u, u)_{N,\omega}, \\ (u, v)_{N,h,\omega} &= \textstyle\int_0^1 (u, v)_{N,\omega} dy, \qquad \|u\|^2_{N,h,\omega} = (u, u)_{N,h,\omega}.\end{aligned}$$

Clearly, if $uv \in \mathcal{P}_{2N-1}(I_x) \otimes L^2(I_y)$, then $(u, v)_{N,h,\omega} = (u, v)_\omega$. Let $P_c : \ C(\bar{I}_x) \to \mathcal{P}_N(\bar{I}_x)$ be the interpolation, i.e., $P_c u(x^{(j)}) = u(x^{(j)})$, for $0 \leq j \leq N$. Let $\Pi^k_h : \ C(\bar{I}_y) \to \tilde{S}^k_h(I_y) \bigcap H^1(I_y)$ be the piecewise Lagrange interpolation of degree k over each I_l. Furthermore, let $P_{N,h} : \ L^2_\omega(\Omega) \to X^K_{N,h}(\Omega)$ be the $L^2_\omega(\Omega)$-orthogonal projection.

Let τ be the step size of time t, $\dot{R}_\tau = \{t = l\tau \ / \ 1 \leq l \leq [\frac{T}{\tau}]\}$ and $R_\tau = \dot{R}_\tau \bigcup \{0\}$. Set

$$u_t(t) = \frac{1}{\tau}(u(t + \tau) - u(t)).$$

For approximating the terms in (2.2), we define

$$d_c(u,v) = \partial_x P_c(v_1 u) + \partial_y P_c(v_2 u), \qquad a_{N,h,\omega}(u,v) = -(\partial_x^2 u, v)_{N,h,\omega} + (\partial_y u, \partial_y v)_{N,h,\omega},$$

$$\Phi_c(u) = 2[P_c(\partial_y u_1 \partial_x u_2) - P_c(\partial_x u_1 \partial_y u_2)].$$

Let $\lambda \geq 0$, $0 \leq \sigma \leq 1$, u and p denote the approximations to U and P respectively. The Chebyshev pseudospectral-finite element scheme for (2.2) is to find $u \in (X_{N,h}^{k+\lambda}(\Omega))^2$ and $p \in Y_{N,h}^k(\Omega)$ such that for all $t \in R_\tau$,

$$\begin{cases} (u_t + d_c(u,u) + \nabla p, v)_{N,h,\omega} + \nu a_{N,h,\omega}(u + \sigma\tau u_t, v) \\ = (f,v)_{N,h,\omega}, & \forall v \in (X_{N,h}^{k+\lambda}(\Omega))^2, \\ a_{N,h,\omega}(p,w) = (\Phi_c(u) - \nabla\cdot f, w)_{N,h,\omega}, & \forall w \in \tilde{X}_{N,h}^k(\Omega), \\ u(0) = P_{N,h}U_0. \end{cases} \tag{2.3}$$

3 Numerical Results.

We take the test functions

$$\begin{aligned} U_1(x,y,t) &= Ae^{Bt}(x^2-1)^2 y(y-1)(2y-1), \\ U_2(x,y,t) &= -2Ae^{Bt}(x^3-x)y^2(y-1)^2, \\ P(x,y,t) &= 4Ae^{2Bt}(x^3-3x)(2y^3-3y^2+0.5). \end{aligned}$$

We use scheme(2.3), in which the interval I_y is divided with the mesh size $h_y = 1/M$. For comparision, we also consider the finite element scheme (FEM). In this case, the domain is divided uniformly into rectangular subdomains with the length $h_x = 2/N^*$ in x–direction and $h_y = 1/M$ in y–direction, U is approximated by quadratic finite element and P by linear finite element. For describing the errors, let

$$\begin{aligned} &\hat{I}_x = \{x_j \ / \ x_j = \cos\tfrac{j\pi}{N}, \ \ 1 \leq j \leq N-1\}, && \text{for} \ \ (2.3), \\ &\hat{I}_x = \{x_j \ / \ x_j = -1 + jh_x, \ \ 1 \leq j \leq N^*-1\}, && \text{for} \ \ \text{FEM}, \\ &\hat{I}_y = \{y_j \ / \ y_j = jh_y, \ \ 1 \leq j \leq M-1\}, && \text{for} \ \ (2.3) \ \ \text{and} \ \ \text{FEM} \end{aligned}$$

and

$$E(U(t)) = \left(\frac{\sum_{q=1}^{2} \sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} |U_q(x,y,t) - u_q(x,y,t)|^2}{\sum_{q=1}^{2} \sum_{x\in\hat{I}_x} \sum_{y\in\hat{I}_y} |U_q(x,y,t)|^2} \right)^{1/2}.$$

The error $E(P(t))$ is defined similarly. In calculations, $\nu = 0.0001, A = 0.2,\ B = 0.1$ and $k = 1$. We first take $N = 4,\ M = N^* = 10,\ \tau = 0.005$ and $\sigma = 0$. The numerical results are shown in the Table I. Clearly, scheme (2.3) gives better results than scheme FEM. We also take $N = 10,\ M = 10,\ \tau = 0.001,\ \sigma = 0$ and $\lambda = 1$ in scheme (2.3). The corresponding results are shown in Table II. We find that when N increases and τ decreases, the better results follow. It shows the convergence of scheme (2.3).

Table I. The errors of scheme (2.3) and FEM.

	Scheme (2.3), $\lambda = 1$		Scheme FEM	
t	$E(U(t))$	$E(P(t))$	$E(U(t))$	$E(P(t))$
0.5	0.9167E-04	0.8455E-03	0.1371E-02	0.6378E-02
1.0	0.1806E-03	0.8727E-03	0.2758E-02	0.6697E-02
1.5	0.2699E-03	0.8969E-03	0.4148E-02	0.7035E-02
2.0	0.3598E-03	0.9229E-03	0.5542E-02	0.7381E-02
2.5	0.4507E-03	0.9500E-03	0.6943E-02	0.7738E-02

Table II. The errors of scheme (2.3).

t	$E(U(t))$	$E(P(t))$
0.5	0.5144E-04	0.2061E-03
1.0	0.7531E-04	0.2120E-03
1.5	0.9928E-04	0.2173E-03
2.0	0.1234E-03	0.2230E-03
2.5	0.1478E-03	0.2288E-03

4 Some Lemmas.

We first introduce some Sobolev spaces with the weight $\omega(x)$. For any integer $r \geq 0$, set

$$|u|_{r,\omega,I_x} = \|\partial_x^r u\|_{\omega,I_x}, \qquad \|u\|_{r,\omega,I_x} = (\sum_{m=0}^{r} |u|^2_{m,\omega,I_x})^{1/2},$$

$$H^r_\omega(I_x) = \{u(x)\ /\ \|u\|_{r,\omega,I_x} < \infty\}.$$

For real $r > 0$, the space $H^r_\omega(I_x)$ is defined by the space interpolation. Furthermore,

$$C_0^\infty(I_x) = \{u(x)\ /\ u \text{ is infinitely differentiable and has a compact support in } I_x\}.$$

Denote by $H^r_{0,\omega}(I_x)$ the closure of $C_0^\infty(I_x)$ in $H^r_\omega(I_x)$. Besides, $L^\infty(I_x)$ is the space of essentially bounded functions with the norm $\|\cdot\|_{\infty,I_x}$.

Next, let B be a Banach space with the norm $\|\cdot\|_B$. Define

$$L^2(D,B)=\{u(z):D\to B\ /\ u \text{ is strongly measurable and } \|u\|_{L^2(I,B)}<\infty\},$$

$$C(D,B)=\{u(z):D\to B\ /\ u \text{ is strongly continuous and } \|u\|_{C(I,B)}<\infty\}$$

where

$$\|u\|_{L^2(D,B)}=(\int_D \|u(z)\|^2_B dz)^{1/2},\quad \|u\|_{C(D,B)}=\max_{z\in D}\|u(z)\|_B.$$

For any integer $s\geq 0$, define

$$H^s(D,B)=\{u(z)\ /\ \|u\|_{H^s(D,B)}<\infty\}$$

equipped with the semi-norm and norm

$$|u|_{H^s(D,B)}=\|\partial_z^s u\|_{L^2(D,B)},\quad \|u\|_{H^s(D,B)}=(\sum_{m=0}^{s}|u|^2_{H^m(D,B)})^{1/2}.$$

For real $s\geq 0$, the space $H^s(D,B)$ is defined by the space interpolation.

We now introduce the non-isotropic space

$$H^{r,s}_\omega(\Omega)=L^2(I_y,H^r_\omega(I_x))\bigcap H^s(I_y,L^2_\omega(I_x)),\quad r,s\geq 0$$

with the norm

$$\|u\|^2_{H^{r,s}_\omega(\Omega)}=\|u\|^2_{L^2(I_y,H^r_\omega(I_x))}+\|u\|^2_{H^s(I_y,L^2_\omega(I_x))}.$$

Also let

$$M^{r,s}_\omega(\Omega)=H^{r,s}_\omega(\Omega)\bigcap H^{s-1}(I_y,H^1_\omega(I_x)),\quad r,s\geq 1,$$
$$A^{r,s}_\omega(\Omega)=H^{\frac{7}{6}}(I_y,H^1_\omega(I_x))\bigcap H^s(I_y,H^{r+1}_\omega(I_x))\bigcap H^{s+1}(I_y,H^r_\omega(I_x)),\quad r,s\geq 0,$$
$$Y^{r,s,\delta}_\omega(\Omega)=H^{r,s}_\omega(\Omega)\bigcap H^{\frac12+\delta}(I_y,H^{[\frac{r}{2}]+\frac12+\delta}_\omega(I_x))\bigcap H^{[\frac{s}{2}]+\frac12+\delta}(I_y,H^{\frac12+\delta}_\omega(I_x)),\quad r,s\geq 0,\ \delta>0,$$
$$\begin{aligned}Y^{r,s,\delta}_{1,\omega}(\Omega)=&H^{r+1,s+1}_\omega(\Omega)\bigcap H^1(I_y,H^r_\omega(I_x))\bigcap H^s(I_y,H^1_\omega(I_x))\bigcap H^{\frac12+\delta}(I_y,H^{[\frac{r}{2}]+\frac32+\delta}_\omega(I_x))\\ &\bigcap H^{[\frac{s}{2}]+\frac12+\delta}(I_y,H^{\frac32+\delta}_\omega(I_x))\bigcap H^{\frac32+\delta}(I_y,H^{[\frac{r}{2}]+\frac12+\delta}_\omega(I_x))\\ &\bigcap H^{[\frac{s}{2}]+\frac32+\delta}(I_y,H^{\frac12+\delta}_\omega(I_x)),\quad r,s\geq 0,\ \delta>0.\end{aligned}$$

Their norms are defined similarly. Furthermore let $H^{r,s}_{0,\omega}(\Omega)$ be the closure of $C_0^\infty(\Omega)$ in $H^{r,s}_\omega(\Omega)$. If $r=s$, then $H^{r,r}_\omega(\Omega)=H^r_\omega(\Omega)$, and denote their semi-norm and norm by $|\cdot|_{r,\omega}$ and $\|\cdot\|_{r,\omega}$ respectively, etc.. Denote by $L^\infty(I_x)$, $L^\infty(\Omega)$ and $W^{1,\infty}(\Omega)$ the usual Sobolev spaces with the norms $\|\cdot\|_{\infty,Ix}$, $\|\cdot\|_\infty$ and $\|\cdot\|_{1,\infty}$, etc..

For simplicity of statements, let $\bar{s} = \min(s, k+1)$ and denote by c a generic positive constant independent of h, N, τ and any function. In some lemmas, we require that there exist a suitably big positive constant c_1 and a positive constant c_2 such that

$$c_1 h^{-\frac{4}{3}} \le N \le c_2 h^{-\frac{4}{3}}. \tag{4.1}$$

Lemma 1. If $u(x,y,t) \in C(\bar{I}_x) \times L^2(I_y)$ for all $t \in R_\tau$, then

$$2(u(t), u_t(t))_{N,h,\omega} = (\|u(t)\|^2_{N,h,\omega})_t - \tau \|u_t(t)\|^2_{N,h,\omega}.$$

Let P_N denote the Chebyshev truncated operator. We have that (see [4])

$$\|u - P_N u\|_{L^q_\omega(I_x)} \le c\sigma_{N,q} N^{-m} \|u\|_{W^{m,q}_\omega(I_x)}, \quad m \ge 0,\ 1 \le q \le \infty \tag{4.2}$$

where $\sigma_{N,q} = 1 + \ln N$ for $q = 1$ or $q = \infty$, and $\sigma_{N,q} = 1$, otherwise.

Lemma 2(Lemma 1 of [1]). If $u \in C(\bar{I}_x) \times L^2(I_y)$ and $v \in \mathcal{P}_N(I_x) \times L^2(I_y)$, then

$$\|v\|_\omega \le \|v\|_{N,h,\omega} \le \sqrt{2}\|v\|_\omega,$$

$$|(u,v)_{N,h,\omega} - (u,v)_\omega| \le c(\|u - P_{N-1}u\|_\omega + \|u - P_c u\|_\omega)\|v\|_\omega.$$

Lemma 3(Lemma 2 of [1]). For any $u \in X^k_{N,h}(\Omega)$ with $k \ge 1$,

$$a_{N,h,\omega}(u,u) \ge \frac{1}{4}\|u\|^2_{1,\omega}.$$

Lemma 4(Lemma 5 of [5]). For any $u \in \tilde{X}^k_{N,h}(\Omega)$ with $k \ge 1$,

$$\|u\|_\infty \le c(N/h)^{1/2}\|u\|_\omega.$$

Moreover, for any $u \in H^1_\omega(\Omega)$,

$$\|P_N u\|_\infty \le c(\ln N)^{1/2}\|u\|_{1,\omega}.$$

Lemma 5(Lemma 2 of [5]). Let $u \in H^{r,1}_{0,\omega}(\Omega) \bigcap H^{r,s}_\omega(\Omega)$ with $0 \le r \le 1$, $s \ge 0$, or $u \in H^1_{0,\omega}(\Omega) \bigcap H^{r,s}_\omega(\Omega)$ with $r > 1$, $s \ge 0$. Then

$$\|u - P_{N,h}u\|_\omega \le c(N^{-r} + h^{\bar{s}})\|u||_{H^{r,\bar{s}}_\omega(\Omega)}.$$

Lemma 6(Lemma 5 of [1]). If $u \in H^\beta(I_y, H^r_\omega(I_x))$ with $\beta \ge 0, r > 1/2$ and $0 \le \alpha \le r$, then

$$\|u - P_c u\|_{H^\beta(I_y, H^\alpha_\omega(I_x))} \le cN^{2\alpha - r}\|u\|_{H^\beta(I_y, H^r_\omega(I_x))}.$$

We now introduce the projection $P^*_{N,h} : \ H^1_{0,\omega}(\Omega) \to X^k_{N,h}(\Omega)$ such that for any $u \in H^1_{0,\omega}(\Omega)$,

$$a_\omega(P^*_{N,h}u, v) = a_\omega(u,v), \quad \forall v \in X^k_{N,h}(\Omega),$$

and the projection $P^c_{N,h}:\ H^1_{0,\omega}(\Omega)\to X^k_{N,h}(\Omega)$ such that for any $u\in H^1_{0,\omega}(\Omega)$,

$$a_{N,h,\omega}(P^c_{N,h}u,v)=a_\omega(u,v),\qquad \forall v\in X^k_{N,h}(\Omega).$$

Lemma 7. Let (4.1) hold and $k\geq 1$. If $u\in H^1_{0,\omega}(\Omega)\bigcap M^{r,s}_\omega(\Omega)$ with $r,s\geq 1$, then

$$\|u-P^*_{N,h}u\|_{1,\omega}\leq c(N^{1-r}+h^{\bar{s}-1})\|u\|_{M^{r,\bar{s}}_\omega(\Omega)},$$

$$\|u-P^c_{N,h}u\|_{1,\omega}\leq c(N^{1-r}+h^{\bar{s}-1})\|u\|_{M^{r,\bar{s}}_\omega(\Omega)}.$$

If in addition $u\in M^{r+\frac{1}{4},s}_\omega(\Omega)$, then

$$\|u-P^*_{N,h}u\|_{\omega}\leq c(N^{-r}+h^{\bar{s}})\|u\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)},$$

$$\|u-P^c_{N,h}u\|_{\omega}\leq c(N^{-r}+h^{\bar{s}})\|u\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)}.$$

Proof. The first and the third conclusions come from Lemma 4 of [6]. Next, as in the proof of lemma 7 of [1], we have

$$\|u-P^c_{N,h}u\|_{1,\omega}\leq c(N^{1-r}+h^{\bar{s}-1})\|u\|_{M^{r,\bar{s}}_\omega(\Omega)}.$$

Finally, following the same line as in the proof of Lemma 7 of [1], we get

$$\|u-P^c_{N,h}u\|_{\omega}\leq c(N^{-1}+h)(N^{\frac{3}{4}-r}+h^{\bar{s}-1})\|u\|_{M^{r+\frac{1}{4},\bar{s}}_\omega(\Omega)}.$$

Lemma 8. Let (4.1) hold. If $u\in H^1_{0,\omega}(\Omega)\bigcap H^\beta(I_y,H^\alpha_\omega(I_x))$ with $\alpha,\beta>1/2$, then

$$\|P^c_{N,h}u\|_\infty\leq c\|u\|_{M^{\frac{9}{8},\frac{7}{6}}_\omega(\Omega)\bigcap H^\beta(I_y,H^\alpha_\omega(I_x))}.$$

If, in addition, $u\in A^{\alpha,\beta}_\omega(\Omega)$ with $\alpha,\beta>1/2$, then

$$\|P^c_{N,h}u\|_{1,\infty}\leq c\|u\|_{A^{\alpha,\beta}_\omega(\Omega)}.$$

Proof. We have

$$\|P^c_{N,h}u\|_\infty\leq\|P^c_{N,h}u-\Pi^k_hP_Nu\|_\infty+\|\Pi^k_hP_Nu\|_\infty.$$

By Lemma 4, Lemma 7, (4.2) and Theorem 3.2.1 of [7],

$$\|P^c_{N,h}u-\Pi^k_hP_Nu\|_\infty\leq c\|u\|_{M^{\frac{9}{8},\frac{7}{6}}_\omega(\Omega)}.$$

By Lemma 5 of [6] and Theorem 3.1.5 of [7],

$$\|\Pi^k_hP_Nu\|_\infty\leq c\|P_Nu\|_{H^\beta(I_y,C(I_x))}\leq c\|u\|_{H^\beta(I_y,H^\alpha_\omega(I_x))}.$$

Next, we prove the second conclusion. Let $P_h^1:\ H_0^1(I_y)\to S_h^k(I_y)$ be such that for any $u\in H_0^1(I_y)$,

$$(\partial_y(P_h^1u-u),\partial_y v)_{L^2(I_y)}=0,\qquad \forall v\in S_h^k(I_y).$$

Then we have (see [8])

$$\|u-P_h^1u\|_{\mu,I_y}\le ch^{\bar{s}-\mu}|u|_{\bar{s},I_y},\qquad s\ge 1,\ 0\le\mu\le 1.$$

Also let $P_N^1:\ H_{0,\omega}^1(I_x)\to V_N(I_x)$ be such that for any $u\in H_{0,\omega}^1(I_x)$,

$$(\partial_x(P_N^1u-u),\partial_x(v\omega))_{L^2(I_x)}=0,\qquad \forall v\in V_N(I_x).$$

We have (see (9.5.17) of [4]),

$$\|u-P_N^1u\|_{\mu,\omega,I_x}\le cN^{\mu-r}\|u\|_{H_\omega^r(I_x)},\qquad 0\le\mu\le 1,\ r\ge\mu.$$

Now, we begin to estimate $\|P_{N,h}^c u\|_{1,\infty}$. In fact,

$$|P_{N,h}^c u|_{1,\infty}\le |P_{N,h}^c u-P_{N,h}^* u|_{1,\infty}+|P_{N,h}^* u|_{1,\infty}. \tag{4.3}$$

By Lemma 4,

$$|P_{N,h}^c u-P_{N,h}^* u|_{1,\infty}\le c\sqrt{\frac{N}{h}}|P_{N,h}^c u-P_{N,h}^* u|_{1,\omega}. \tag{4.4}$$

Let ϑ be the identity operator. By Lemma 3 and the definitions of $P_{N,h}^c$ and $P_{N,h}^*$,

$$\begin{aligned}||P_{N,h}^c u-P_{N,h}^* u||_{1,\omega}^2 \le\ & 4a_{N,h,\omega}(P_{N,h}^c u-P_{N,h}^* u,P_{N,h}^c u-P_{N,h}^* u)\\ \le\ & 4|((\vartheta-P_{N-1})\partial_y P_{N,h}^* u,\partial_y(P_{N,h}^c u-P_{N,h}^* u))_\omega|.\end{aligned}$$

Therefore

$$\|P_{N,h}^c u-P_{N,h}^* u\|_{1,\omega}\le c||(\vartheta-P_{N-1})(\partial_y P_{N,h}^* u-\partial_y P_h^1P_N^1u)\|_\omega+c\|(\vartheta-P_{N-1})\partial_y P_h^1P_N^1u\|_\omega. \tag{4.5}$$

Furthermore

$$\|(\vartheta-P_{N-1})(\partial_y P_{N,h}^* u-\partial_y P_h^1P_N^1u)\|_\omega\le c\|P_{N,h}^* u-P_h^1P_N^1u\|_{1,\omega}. \tag{4.6}$$

Let $W=P_{N,h}^* u-P_h^1P_N^1u$. We have from Lemma 3 and the definition of $P_{N,h}^*$ that

$$\begin{aligned}\|P_{N,h}^* u-P_h^1P_N^1u\|_{1,\omega}^2\le\ & 4a_\omega(P_{N,h}^* u-P_h^1P_N^1u,P_{N,h}^* u-P_h^1P_N^1u)\\ =\ & 4(\partial_y(u-P_h^1P_N^1u),\partial_y W)_\omega+4(\partial_x(u-P_h^1P_N^1u),\partial_x(W\omega))_{L^2(\Omega)}\\ =\ & 4(\partial_y(u-P_N^1u),\partial_y W)_\omega+4(\partial_x(u-P_h^1u),\partial_x(W\omega))_{L^2(\Omega)}.\end{aligned}$$

Thus

$$\|P^*_{N,h}u - P^1_h P^1_N u\|_{1,\omega} \leq \ c(N^{-r} + h^{\bar{s}})\|u\|_{H^1(I_y,H^r_\omega(I_x))\bigcap H^{\bar{s}}(I_y,H^1_\omega(I_x))}. \tag{4.7}$$

Similarly

$$\|(\vartheta - P_{N-1})\partial_y P^1_h P^1_N u\|_\omega \leq cN^{-r}\|u\|_{H^1(I_y,H^r_\omega(I_x))}. \tag{4.8}$$

Thus we have from (4.5)–(4.8) that

$$\|P^c_{N,h}u - P^*_{N,h}u\|_{1,\omega} \leq c(N^{-r} + h^{\bar{s}})\|u\|_{H^1(I_y,H^r_\omega(I_x))\bigcap H^{\bar{s}}(I_y,H^1_\omega(I_x))}$$

and so (4.4) implies that

$$|P^c_{N,h}u - P^*_{N,h}u|_{1,\infty} \leq c\|u\|_{H^1(I_y,H^{\frac{7}{8}}_\omega(I_x))\bigcap H^{\frac{7}{6}}(I_y,H^1_\omega(I_x))}. \tag{4.9}$$

Now, we turn to estimate $|P^*_{N,h}u|_{1,\infty}$. Clearly

$$|P^*_{N,h}u|_{1,\infty} \leq c\sqrt{\frac{N}{h}}|P^*_{N,h}u - P^1_h P^1_N u|_{1,\omega} + |P^1_h P^1_N u|_{1,\infty}. \tag{4.10}$$

From (4.7),

$$\sqrt{\frac{N}{h}}|P^*_{N,h}u - P^1_h P^1_N u|_{1,\omega} \leq c\|u\|_{H^1(I_y,H^{\frac{7}{8}}_\omega(I_x))\bigcap H^{\frac{7}{6}}(I_y,H^1_\omega(I_x))}. \tag{4.11}$$

On the other hand,

$$|P^1_h P^1_N u|_{1,\infty} \leq \|\partial_y P^1_h P^1_N u\|_\infty + \|\partial_x P^1_h P^1_N u\|_\infty.$$

Furthermore

$$\begin{aligned} \|\partial_y P^1_h P^1_N u\|_\infty \leq \ & \|\partial_y P^1_h P^1_N u - \partial_y P^1_h P_N u\|_\infty + \|\partial_y P^1_h P_N u - \partial_y \Pi^k_h P_N u\|_\infty + \|\partial_y \Pi^k_h P_N u\|_\infty \\ \leq \ & c\|u\|_{H^1(I_y,H^{\frac{7}{8}}_\omega(I_x))\bigcap H^{\beta+1}(I_y,H^\alpha_\omega(I_x))}. \end{aligned} \tag{4.12}$$

Similarly

$$\|\partial_x P^1_h P^1_N u\|_\infty \leq c\|u\|_{H^{\frac{7}{6}}(I_y,H^1_\omega(I_x))\bigcap H^\beta(I_y,H^{\alpha+1}_\omega(I_x))} \tag{4.13}$$

and so

$$\|P^1_h P^1_N u\|_{1,\infty} \leq c\|u\|_{H^1(I_y,H^{\frac{7}{8}}_\omega(I_x))\bigcap H^{\frac{7}{6}}(I_y,H^1_\omega(I_x))\bigcap H^{\beta+1}(I_y,H^\alpha_\omega(I_x))\bigcap H^\beta(I_y,H^{\alpha+1}_\omega(I_x))}. \tag{4.14}$$

Therefore we have from (4.10), (4.11) and (4.14) that $|P^*_{N,h}u|_{1,\infty} \leq c\|u\|_{A^{\alpha,\beta}_\omega(\Omega)}$.

Lemma 9(Lemma 9 of [1]). If $u, v \in Y^{r,s,\delta}_\omega(\Omega)$ with $r, s \geq 0$ and $\delta > 0$, then

$$\|uv\|_{H^{r,s}_\omega(\Omega)} \leq c\|u\|_{Y^{r,s,\delta}_\omega(\Omega)}\|v\|_{Y^{r,s,\delta}_\omega(\Omega)}.$$

Lemma 10(Lemma 10 of [1]). If $u \in L^2_\omega(\Omega)$ and $v \in H^1_{0,\omega}(\Omega)$, then

$$|(u, \partial_x(v\omega))_{L^2(\Omega)}| \leq 2\|u\|_\omega |v|_{1,\omega}.$$

Lemma 11(Lemma 8 of [6]). There exists a positive constant c_d depending only on the value of d, such that for all $u \in \mathcal{P}_N(I_x) \otimes (H^1(I_y) \bigcap \tilde{S}^k_h(I_y))$,

$$|u|^2_{1,\omega} \leq (2N^4 + c_d h^{-2})\|u\|^2_\omega.$$

Lemma 12. Let (4.1) hold. $u \in W_N(I_x) \times (\tilde{S}^k_h(I_y) \bigcap H^1(I_y))$ and $g \in \mathcal{P}_N(I_x) \times (\tilde{S}^k_h(I_y) \bigcap H^1(I_y))$ satisfy the following equation

$$a_{N,h,\omega}(u, v) = (g, v)_{N,h,\omega}, \qquad \forall v \in \tilde{X}^k_{N,h}(\Omega). \tag{4.15}$$

Then we have

$$\|u\|_{1,\omega} \leq c\|g\|_\omega.$$

Proof. Let $\eta = (1 - x^2)^{1/2}$. Define the spaces $L^2_\eta(I_x)$ with the norm $\|u\|_{\eta,I_x}$ and $H^r_\eta(I_x)$ with the norm $\|u\|_{r,\eta,I_x}$ in the same way as $L^2_\omega(I_x)$ and $H^r_\omega(I_x)$, etc.. From Lemma 3.1 of [9], for any $\xi \in L^2_\eta(I_x)$ and $\lambda > 0$, there exists a unique function $W \in H^2_\eta(I_x)$ such that

$$\begin{cases} LW = -\frac{d^2W}{dx^2} + \lambda W = \xi, & \text{in } I_x, \\ \frac{dW}{dx}(-1) = \frac{dW}{dx}(1) = 0. \end{cases} \tag{4.16}$$

Let $H^{-s}(I_x)$ be the dual space of $H^s(I_x)$. We have from (4.16) that

$$\|W\|^2_{H^1(I_x)} + \lambda\|W\|^2_{L^2(I_x)} \leq c\|\xi\|^2_{H^{-1}(I_x)}.$$

Furthermore, if $\xi \in L^2(I_x)$, then by multiplying (4.16) by $\frac{d^2W}{dx^2}$ and integrating by parts,

$$\|W\|^2_{H^2(I_x)} + \lambda\|W\|^2_{H^1(I_x)} \leq c\|\xi\|^2_{L^2(I_x)}.$$

By the space interpolation, if $\xi \in H^{-s}(I_x)$ with $0 \leq s \leq 1$, then $W \in H^{2-s}(I_x)$ and

$$\|W\|^2_{H^{2-s}(I_x)} + \lambda\|W\|^2_{H^{1-s}(I_x)} \leq c\|\xi\|^2_{H^{-s}(I_x)}. \tag{4.17}$$

Since for any real $s \geq 0$, $H^s_\eta(I_x) \subset H^{s-1/4}(I_x)$(see Theorem 4.2 of [9]), we get

$$\|W\|^2_{H^{\frac{7}{4}}(I_x)} + \lambda\|W\|^2_{H^{\frac{3}{4}}(I_x)} \leq c\|\xi\|^2_{H^{-\frac{1}{4}}(I_x)} \leq c\|\xi\|^2_{\eta,I_x}. \tag{4.18}$$

Moreover for any real $s \geq 1/4$, $H^s(I_x) \subset H^{s-1/4}_\omega(I_x)$(see Theorem 4.1 of [9]), and thus

$$\|W\|^2_{\frac{3}{2},\omega,I_x} + \lambda\|W\|^2_{\frac{1}{2},\omega,I_x} \leq c\|\xi\|^2_{\eta,I_x} = c\|LW\|^2_{\eta,I_x}. \tag{4.19}$$

Next, consider an auxiliary problem. It is to find $\lambda \in \mathcal{R}$ and $\phi \in \tilde{S}_h^k(I_y) \bigcap H^1(I_y)$, such that

$$\int_0^1 \frac{d\phi}{dy}\frac{dz}{dy}dy = \lambda \int_0^1 \phi z dy, \qquad \forall z \in \tilde{S}_h^k(I_y) \bigcap H^1(I_y). \tag{4.20}$$

Then there exists a normalized $L^2(I_y)$-orthogonal eigenfunction system $\{\phi_l(y)\}$, $l = 0, 1, 2, \cdots, M_h$. The corresponding eigenvalues are ranged as

$$0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots \le \lambda_{M_h}.$$

Let $\phi_0(y) \equiv 1$. By (4.20), we know that

$$\int_0^1 \frac{d\phi_k}{dy}\frac{d\phi_l}{dy}dy = \lambda_k \delta_{kl}, \qquad k, l = 0, 1, 2, \cdots, M_h. \tag{4.21}$$

Let $\{\lambda^{(l)}\}$ be the eigenvalues of the corresponding continuous problem, that is

$$\begin{cases} -\dfrac{d^2u}{dy^2} = \lambda u, \\ \dfrac{du}{dy}(0) = \dfrac{du}{dy}(1) = 0. \end{cases}$$

Then $\lambda^{(l)} = \pi^2 l^2$, $l = 0, 1, \cdots$. We have that for sufficiently small h and certain positive constant δ (see [2]),

$$\lambda^{(l)} \le \lambda_l \le \lambda^{(l)} + 2\delta h^{2k}[\lambda^{(l)}]^{k+1}. \tag{4.22}$$

Now, we turn to prove the lemma. Since $u \in W_N(I_x) \times (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))$ and $g \in \mathcal{P}_N(I_x) \times (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))$, we put

$$u = \sum_{l=0}^{M_h} u_l(x)\phi_l(y), \qquad g = \sum_{l=0}^{M_h} g_l(x)\phi_l(y)$$

where $u_l \in W_N(I_x)$, $g_l \in \mathcal{P}_N(I_x)$. From (4.15),

$$-(\sum_{l=0}^{M_h} \frac{d^2u_l}{dx^2}\phi_l(y), v)_\omega + (\sum_{l=0}^{M_h} u_l(x)\partial_y\phi_l(y), \partial_y v)_{N,h,\omega} = (\sum_{l=0}^{M_h} g_l(x)\phi_l(y), v)_{N,h,\omega}, \ \forall v \in \tilde{X}_{N,h}^k(\Omega). \tag{4.23}$$

We first let $l \ne 0$, $v = (1 - x^2)z_l(x)\phi_l(y)$ and $z_l(x) \in \mathcal{P}_{N-2}(I_x)$. Then

$$-(\frac{d^2u_l}{dx^2}, z_l)_{\eta,I_x} + \lambda_l(u_l, (1-x^2)z_l)_{N,\omega} = (g_l, (1-x^2)z_l)_{N,\omega}$$

and so

$$(Lu_l, z_l)_{\eta,I_x} \equiv (-\frac{d^2u_l}{dx^2} + \lambda_l u_l, z_l)_{\eta,I_x} = (g_l, z_l)_{\eta,I_x} + E_1(z_l) + E_2(z_l) \tag{4.24}$$

with

$$E_1(z_l) = \lambda_l(u_l,(1-x^2)z_l)_{\omega,I_x} - \lambda_l(u_l,(1-x^2)z_l)_{N,\omega},$$
$$E_2(z_l) = (g_l,(1-x^2)z_l)_{N,\omega} - (g_l,(1-x^2)z_l)_{\omega,I_x}.$$

Let $\tilde{P}_N$ be the $L^2_\omega(I_x)$-orthogonal projection, and $z_l = \tilde{P}_{N-2}Lu_l$. Then we have

$$z_l = Lu_l - \lambda_l(u_l - \tilde{P}_{N-2}u_l) \in \mathcal{P}_{N-2}(I_x).$$

So (4.24) reads

$$\begin{aligned}\|Lu_l\|^2_{\eta,I_x} - \lambda_l(Lu_l, u_l - \tilde{P}_{N-2}u_l)_{\eta,I_x} = \; & (g_l, Lu_l)_{\eta,I_x} - \lambda_l(g_l, u_l - \tilde{P}_{N-2}u_l)_{\eta,I_x} \\ & + E_1(\tilde{P}_{N-2}Lu_l) + E_2(\tilde{P}_{N-2}Lu_l).\end{aligned} \tag{4.25}$$

Furthermore, since $(1-x^2)z_l \in \mathcal{P}_N(I_x)$, we get from (9.3.5) of [4] that

$$\begin{aligned}|E_1(\tilde{P}_{N-2}Lu_l)| \le \; & c\lambda_l\|u_l - \tilde{P}_{N-1}u_l\|_{\omega,I_x}(\|Lu_l\|_{\eta,I_x} + \lambda_l\|u_l - \tilde{P}_{N-2}u_l\|_{\eta,I_x}), \\ |E_2(\tilde{P}_{N-2}Lu_l)| \le \; & c\|g_l\|_{\omega,I_x}(\|Lu_l\|_{\eta,I_x} + \lambda_l\|u_l - \tilde{P}_{N-2}u_l\|_{\eta,I_x}).\end{aligned}$$

By substituting the above estimations into (4.25), we obtain

$$\frac{1}{2}\|Lu_l\|^2_{\eta,I_x} \le c\|g_l\|^2_{\omega,I_x} + c\lambda_l^2(\|u_l - \tilde{P}_{N-1}u_l\|^2_{\omega,I_x} + \|u_l - \tilde{P}_{N-2}u_l\|^2_{\eta,I_x}). \tag{4.26}$$

We know from (4.19) that

$$\|u_l\|^2_{\frac{3}{2},\omega,I_x} + \lambda_l\|u_l\|^2_{\frac{1}{2},\omega,I_x} \le c\|Lu_l\|^2_{\eta,I_x}.$$

Since

$$\|u_l - \tilde{P}_{N-1}u_l\|_{\omega,I_x} \le cN^{-r}\|u_l\|_{r,w,I_x}, \quad \text{for } r \ge 0,$$

we obtain from (4.26) that

$$\|u_l\|^2_{\frac{3}{2},\omega,I_x} + \lambda_l\|u_l\|^2_{\frac{1}{2},\omega,I_x} \le c\|g_l\|^2_{\omega,I_x} + c\lambda_l^2 N^{-3}\|u_l\|^2_{\frac{3}{2},\omega,I_x}. \tag{4.27}$$

We know from (4.22) that

$$\lambda_l \le \lambda_{M_h} \le \lambda^{(M_h)} + 2\delta h^{2k}[\lambda^{(M_h)}]^{k+1}, \quad \lambda^{(M_h)} = (M_h)^2\pi^2.$$

Since $M_h = O(\frac{1}{h})$, we have $\lambda_l \le ch^{-2}$. Thus $1 - c\lambda_l^2 N^{-3} \ge 1 - c\lambda_{M_h}^2 N^{-3} \ge 1 - ch^{-4}N^{-3}$. Thanks to condition (4.1), we have $1 - ch^{-4}N^{-3} \ge \alpha > 0$. Hence (4.27) implies that

$$\|u_l\|^2_{\frac{3}{2},\omega,I_x} + \lambda_l\|u_l\|^2_{\frac{1}{2},\omega,I_x} \le c\|g_l\|^2_{\omega,I_x}$$

and so for $l \ne 0$,

$$|u_l|^2_{1,\omega,I_x} + \lambda_l\|u_l\|^2_{\omega,I_x} \le c\|g_l\|^2_{\omega,I_x}. \tag{4.28}$$

Next, we consider the case with $l = 0$. By taking $v = -(1-x^2)\partial_x^2 u_0 \phi_0(y) = -(1-x^2)\partial_x^2 u_0$ in (4.23), we get

$$\|\partial_x^2 u_0\|^2_{\eta,I_x} = -(g_0,(1-x^2)\partial_x^2 u_0)_{N,\omega} \le c\|g_0\|_{\omega,I_x}\|\partial_x^2 u_0\|_{\eta,I_x}.$$

Hence $||\partial_x^2 u_0||_{\eta,I_x} \le c||g_0||_{\omega,I_x}$. On the other hand, since $u_0 \in W_N(I_x)$, we have

$$|u_0|^2_{1,\omega,I_x} \le \int_{I_x}\omega(x)(\int_{I_x}\eta(x)(\partial_x^2 u_0)^2 dx)(\int_{I_x}\omega(x)dx)dx \le c\|\partial_x^2 u_0\|^2_{\eta,I_x}$$

and so

$$|u_0|^2_{1,\omega,I_x} \le c\|g_0\|^2_{\omega,I_x}. \tag{4.29}$$

Finally, we have from (4.28) and (4.29) that

$$\|u\|^2_{1,\omega} = \sum_{l=0}^{M_h}(|u_l|^2_{1,\omega,I_x} + \lambda_l\|u_l\|^2_{\omega,I_x}) \le c\|g\|^2_\omega.$$

5 The Generalized Stability.

Assume that $u(0)$ and $f(t)$ have the errors $\tilde{u}(0)$ and $\tilde{f}(t)$ which induce the errors of $u(t)$ and $p(t)$, denoted by $\tilde{u}(t)$ and $\tilde{p}(t)$ respectively. Then

$$\begin{cases} (\tilde{u}_t, v)_{N,h,\omega} + (d_c(\tilde{u}, u+\tilde{u}) + d_c(u,\tilde{u}) + \nabla\tilde{p}, v)_{N,h,\omega} & \\ +\nu a_{N,h,\omega}(\tilde{u}+\sigma\tau\tilde{u}_t, v) = (\tilde{f}, v)_{N,h,\omega}, & \forall v \in (X^{k+\lambda}_{N,h}(\Omega))^2, \\ a_{N,h,\omega}(\tilde{p}, w) = (\Phi_c(\tilde{u}) + \Phi_c^*(u,\tilde{u}) - \nabla\cdot\tilde{f}, w)_{N,h,\omega}, & \forall w \in \tilde{X}^k_{N,h}(\Omega) \end{cases} \tag{5.1}$$

where

$$\Phi_c^*(u,\tilde{u}) = 2[P_c(\partial_y u_1\partial_x\tilde{u}_2) + P_c(\partial_y\tilde{u}_1\partial_x u_2) - P_c(\partial_x u_1\partial_y\tilde{u}_2) - P_c(\partial_x\tilde{u}_1\partial_y u_2)].$$

Let $\varepsilon > 0$ and m be an undetermined positive constant. By taking $v = 2\tilde{u}(t) + m\tau\tilde{u}_t(t)$ in the first formula of (5.1), we have from Lemma 1, Lemma 2 and Lemma 3 that

$$\begin{aligned} &(\|\tilde{u}\|^2_{N,h,\omega})_t + \tau(m-1-\varepsilon)\|\tilde{u}_t\|^2_{N,h,\omega} + \tfrac{\nu}{2}||\tilde{u}||^2_{1,\omega} + \tfrac{\nu\sigma m\tau^2}{4}\|\tilde{u}_t\|^2_{1,\omega} + \nu\tau(\sigma+\tfrac{m}{2}) \\ &((\|\partial_x\tilde{u}\|^2_{N,h,\omega} + \|\partial_y\tilde{u}\|^2_{N,h,\omega})_t - \tau\|\partial_x\tilde{u}_t\|^2_{N,h,\omega} - \tau\|\partial_y\tilde{u}_t\|^2_{N,h,\omega}) + A + B + \sum_{j=1}^{6}F_j \\ &\le 2\|\tilde{u}\|^2_\omega + (2+\tfrac{\tau m^2}{2\varepsilon})\|P_c\tilde{f}\|^2_\omega \end{aligned} \tag{5.2}$$

where

$$A = 2\nu\sigma\tau(\partial_x\tilde{u}_t, x\omega^2\tilde{u})_\omega, \qquad B = \nu m\tau(\partial_x\tilde{u}, x\omega^2\tilde{u}_t)_\omega,$$

$$F_1 = (d_c(\tilde{u},u), 2\tilde{u}+m\tau\tilde{u}_t)_{N,h,\omega}, \qquad F_2 = (d_c(u,\tilde{u}), 2\tilde{u}+m\tau\tilde{u}_t)_{N,h,\omega},$$

$$F_3 = 2(d_c(\tilde{u},\tilde{u}),\tilde{u})_{N,h,\omega}, \qquad F_4 = m\tau(d_c(\tilde{u},\tilde{u}),\tilde{u}_t)_{N,h,\omega},$$

$$F_5 = 2(\nabla\tilde{p},\tilde{u})_{N,h,\omega}, \qquad F_6 = m\tau(\nabla\tilde{p},\tilde{u}_t)_{N,h,\omega}.$$

We have from Lemma 1 of [10] that $\|v\omega^2\|_{\omega,I_x} \leq |v|_{1,\omega,I_x}$ for any $v \in H^1_{0,\omega}(I_x)$, and so

$$|A| + |B| \leq \frac{\nu}{4}(\sigma + \frac{m}{2})\|\partial_x \tilde{u}\|^2_\omega + 4\nu\tau^2(\sigma + \frac{m}{2})\|\partial_x \tilde{u}_t\|^2_\omega.$$

Let $|u|^2_{1,N,h,\omega} = \|\partial_x u\|^2_{N,h,\omega} + \|\partial_y u\|^2_{N,h,\omega}$. Then (5.2) leads to

$$\begin{aligned}&(\|\tilde{u}\|^2_{N,h,\omega})_t + \tau(m-1-\varepsilon)\|\tilde{u}_t\|^2_{N,h,\omega} + \tfrac{\nu}{8}(4-m-2\sigma)|\tilde{u}|^2_{1,\omega} + \nu\tau(\sigma + \tfrac{m}{2})(|\tilde{u}|^2_{1,N,h,\omega})_t \\ &+ \tfrac{\nu\sigma m\tau^2}{4}\|\tilde{u}_t\|^2_{1,\omega} - 5\nu\tau^2(\sigma + \tfrac{m}{2})|\tilde{u}_t|^2_{1,N,h,\omega} + \sum_{j=1}^{6} F_j \leq 2\|\tilde{u}\|^2_\omega + (2 + \frac{\tau m^2}{4\varepsilon})\|P_c\tilde{f}\|^2_\omega.\end{aligned} \quad (5.3)$$

Now we estimate$|F_j|$. Firstly, by Lemma 2 and Lemma 10,

$$|F_1| \leq \tfrac{\varepsilon\nu}{8}|\tilde{u}|^2_{1,\omega} + \tfrac{\varepsilon\nu m\tau^2}{6}|\tilde{u}_t|^2_{1,\omega} + \tfrac{c(m+1)}{\varepsilon\nu}\|u\|^2_\infty\|\tilde{u}\|^2_\omega.$$

Similarly

$$|F_2| \leq \frac{\varepsilon\nu}{8}|\tilde{u}|^2_{1,\omega} + \frac{\varepsilon\nu m\tau^2}{6}|\tilde{u}_t|^2_{1,\omega} + \frac{c(m+1)}{\varepsilon\nu}\|u\|^2_\infty\|\tilde{u}\|^2_\omega.$$

By Lemma 4 and Lemma 10,

$$|F_3| \leq \tfrac{\varepsilon\nu}{8}|\tilde{u}|^2_{1,\omega} + \tfrac{c\ln N}{\varepsilon\nu}\|\tilde{u}\|^2_\omega|\tilde{u}|^2_{1,\omega}.$$

Similarly

$$|F_4| \leq \tfrac{\varepsilon\nu m\tau^2}{6}|\tilde{u}_t|^2_{1,\omega} + \tfrac{cm\ln N}{\varepsilon\nu}||\tilde{u}||^2_\omega|\tilde{u}|^2_{1,\omega}.$$

Now, we apply Lemma 12 to the second formula of (5.1) and obtain that

$$|\tilde{p}|_{1,\omega} \leq c(\|\Phi_c(\tilde{u})\|_\omega + \|\Phi^*_c(u,\tilde{u})\|_\omega + \|P_c(\nabla\cdot\tilde{f})\|_\omega).$$

Moreover,

$$\begin{aligned}\|\Phi_c(\tilde{u})\|_\omega &\leq c|\tilde{u}|_{1,\infty}|\tilde{u}|_{1,\omega} \leq c\sqrt{\tfrac{N}{h}}|\tilde{u}|^2_{1,\omega}, \\ \|\Phi^*_c(u,\tilde{u})\|_\omega &\leq c|u|_{1,\infty}|\tilde{u}|_{1,\omega}.\end{aligned}$$

Thus

$$|\tilde{p}|_{1,\omega} \leq c(|u|_{1,\infty}|\tilde{u}|_{1,\omega} + \sqrt{\frac{N}{h}}|\tilde{u}|^2_{1,\omega} + \|P_c(\nabla\cdot\tilde{f})\|_\omega)$$

and

$$|F_5| \leq \tfrac{\varepsilon\nu}{8}|\tilde{u}|^2_{1,\omega} + c(1 + \tfrac{1}{\varepsilon\nu}|u|^2_{1,\infty})\|\tilde{u}\|^2_\omega + \tfrac{cN}{\varepsilon\nu h}\|\tilde{u}\|^2_\omega|\tilde{u}|^2_{1,\omega} + c\|P_c(\nabla\cdot\tilde{f})\|^2_\omega.$$

By Lemma 11,

$$\begin{aligned}|F_6| \leq\ & \varepsilon\tau\|\tilde{u}_t\|^2_\omega + \tfrac{cm^2\tau}{\varepsilon}|u|^2_{1,\infty}|\tilde{u}|^2_{1,\omega} + \tfrac{cm^2\tau}{\varepsilon}\|P_c(\nabla\cdot\tilde{f})\|^2_\omega \\ &+ \tfrac{cm\tau N}{\varepsilon h}(2N^4 + c_d h^{-2})\|\tilde{u}\|^2_\omega|\tilde{u}|^2_{1,\omega}.\end{aligned}$$

Let $|||u|||_{1,\infty} = \max_{t\in R_\tau} \|u(t)\|_{1,\infty}$, etc.. By substituting the above estimations into (5.3), we have from Lemma 2 and Lemma 12 that

$$\begin{aligned}(\|\tilde{u}\|^2_{N,h,\omega})_t \quad & +\tau[m-1-2\varepsilon-\nu\tau(5(2\sigma+m)+\tfrac{\varepsilon m}{2}-\tfrac{\sigma m}{4})(2N^4+c_dh^{-2})]\|\tilde{u}_t\|^2_\omega \\ & +\tfrac{\nu}{8}(\tfrac{15}{4}-m-2\sigma-4\varepsilon)|\tilde{u}|^2_{1,\omega}+\nu\tau(\sigma+\tfrac{m}{2})(|\tilde{u}|^2_{1,N,h,\omega})_t \\ \le \quad & M_1\|\tilde{u}\|^2_\omega+B(\|\tilde{u}\|_\omega)|\tilde{u}|^2_{1,\omega}+G_1.\end{aligned} \tag{5.4}$$

where

$$\begin{aligned}M_1 \quad &= c+\tfrac{c}{\varepsilon\nu}[(1+m)|||u|||^2_\infty+|||u|||^2_{1,\infty}], \\ B(\|\tilde{u}\|_\omega) \quad &= -\tfrac{\nu}{32}+\tfrac{cm^2\tau}{\varepsilon}|||u|||^2_{1,\infty} \\ & \quad +[\tfrac{c}{\varepsilon\nu}(1+m)\ln N+\tfrac{cN}{\varepsilon h}(\tfrac{1}{\nu}+m^2\tau(2N^4+c_dh^{-2}))]\|\tilde{u}\|^2_\omega, \\ G_1 \quad &= (2+\tfrac{m^2\tau}{2\varepsilon})\|P_c\tilde{f}\|^2_\omega+c(1+\tfrac{m^2\tau}{\varepsilon})\|P_c(\nabla\cdot\tilde{f})\|^2_\omega.\end{aligned}$$

Let ε be suitably small and μ suitably large. Suppose hat

$$\nu\tau(2N^4+c_dh^{-2}) < \frac{1}{\mu(5+\frac{\varepsilon}{2}-\frac{\sigma}{4})}. \tag{5.5}$$

We take

$$m = (\frac{33}{32}+2\varepsilon+10\sigma\nu\tau(2N^4+c_dh^{-2}))(1-\frac{1}{\mu})^{-1}.$$

Then the coefficient of the term $\|\tilde{u}_t\|^2_\omega$ in (5.4) is not less than $\frac{\tau}{32}$. Moreover, if

$$\mu > (\frac{10\sigma}{5+\frac{\varepsilon}{2}-\frac{\sigma}{4}}+\frac{7}{2}-2\sigma-4\varepsilon)(\frac{79}{32}-2\sigma-6\varepsilon)^{-1}, \tag{5.6}$$

then the coefficient of the term $|\tilde{u}|^2_{1,\omega}$ in (5.4) is not less than $\frac{\nu}{32}$. Thus (5.4) reads

$$\begin{aligned}&(\|\tilde{u}\|^2_{N,h,\omega})_t+\tfrac{\tau}{32}\|\tilde{u}_t\|^2_\omega+\tfrac{\nu}{32}|\tilde{u}|^2_{1,\omega}+\nu\tau(\sigma+\tfrac{m}{2})(|\tilde{u}|^2_{1,N,h,\omega})_t \\ &\le M_1\|\tilde{u}\|^2_\omega+B(\|\tilde{u}\|_\omega)|\tilde{u}|^2_{1,\omega}+G_1.\end{aligned} \tag{5.7}$$

Let

$$\begin{aligned}E(t) &= \|\tilde{u}(t)\|^2_\omega+\tfrac{\tau}{32}\textstyle\sum_{\substack{t'\in R_\tau\\ t'<t}}(\tau\|\tilde{u}_t(t')\|^2_\omega+\nu|\tilde{u}(t')|^2_{1,\omega}), \\ \rho(t) &= 2\|\tilde{u}(0)\|^2_\omega+\nu\tau(2\sigma+m)|\tilde{u}(0)|^2_{1,\omega}+\tau\textstyle\sum_{\substack{t'\in R_\tau\\ t'<t}}G_1(t').\end{aligned}$$

By summing (5.7) for $t\in R_\tau$, we have

$$E(t) \le \rho(t)+\tau\sum_{\substack{t'\in R_\tau\\ t'<t}}(M_1E(t')+B(E(t'))|\tilde{u}|^2_{1,\omega}).$$

Finally, we use a discrete inequality Lemma 4.16 of [2] to get the following conclusion.

Theorem 1. Assume that
(i) (5.5) and (5.6) hold;
(ii) for certain suitably small positive constant c_3, $\tau|||u|||^2_{1,\infty} < c_3\nu$;
(iii) there exist positive constants d_1 and d_2 depending only on $|||u|||_{1,\infty}$ and ν such that $\rho(t_1)e^{d_1 t_1} \leq \frac{d_2 h}{N}$ for some $t_1 \in R_\tau$.
Then for all $t \in R_\tau$, $t \leq t_1$, we have

$$E(t) \leq \rho(t)e^{d_1 t}.$$

6. The Convergence

Let the $\tilde{P}_h^1$ be the operator such that for any $u \in H^1(I_y)$,

$$\int_0^1 \partial_y u \partial_y v dy = \int_0^1 \partial_y(\tilde{P}_h^1 u)\partial_y v dy, \qquad \forall v \in \tilde{S}_h^k(I_y) \bigcap H^1(I_y),$$
$$\int_0^1 (u - \tilde{P}_h^1 u) dy = 0.$$

We have that for any $v \in H^s(I_y)$ with $s \geq 1$,

$$\|u - \tilde{P}_h^1 u\|_{H^\mu(I_y)} \leq ch^{\bar{s}-\mu}\|u\|_{H^s(I_y)}, \qquad 0 \leq \mu \leq 1,\ \mu \leq \bar{s}. \tag{6.1}$$

Let $U^* = P_{N,h}^c U$ and $P^* \in \mathcal{P}_N(I_x) \times (\tilde{S}_h^k(I_y) \bigcap H^1(I_y))$ defined by

$$P^* = \tilde{P}_h^1 P(-1,y) + \int_{-1}^x \tilde{P}_h^1 P_{N-1}^1 \frac{\partial P}{\partial s}(s,y)ds.$$

Set $\tilde{U} = u - U^*$ and $\tilde{P} = p - P^*$. Then by (2.2) and (2.3),

$$\begin{cases} (\tilde{U}_t + d_c(\tilde{U}, U^* + \tilde{U}) + d_c(U^*, \tilde{U}) + \nabla\tilde{P}, v)_{N,h,\omega} + \\ \nu a_{N,h,\omega}(\tilde{U} + \sigma\tau\tilde{U}_t, v) = \sum_{j=1}^5 A_j(v), & \forall v \in (X_{N,h}^{k+\lambda}(\Omega))^2, \\ a_{N,h,\omega}(\tilde{P}, w) = (\Phi_c(\tilde{U}) + \Phi_c^*(U^*, \tilde{U}), w)_{N,h,\omega} + \sum_{j=6}^8 A_j(w), & \forall w \in \tilde{X}_{N,h}^k(\Omega), \\ \tilde{U}(0) = P_{N,h}U_0 - P_{N,h}^c U_0, \end{cases}$$

where

$$A_1(v) = (\partial_t U, v)_\omega - (U_t^*, v)_{N,h,\omega}, \qquad A_2(v) = (d(U,U), v)_\omega - (d_c(U^*, U^*), v)_{N,h,\omega},$$
$$A_3(v) = -\nu\sigma\tau a_{N,h,\omega}(U_t^*, v), \qquad A_4(v) = (\nabla P, v)_\omega - (\nabla P^*, v)_{N,h,\omega},$$
$$A_5(v) = (f, v)_{N,h,\omega} - (f, v)_\omega, \qquad A_6(w) = a_\omega(P, w) - a_{N,h,\omega}(P^*, w),$$
$$A_7(w) = -(\Phi(U), w)_\omega + (\Phi_c(U^*), w)_{N,h,\omega}, \quad A_8(w) = (\nabla \cdot f, w)_\omega - (\nabla \cdot f, w)_{N,h,\omega}.$$

Now, we turn to estimate $|A_j|$. Let $\bar{s}$ be the same as before and $\tilde{s}=\min(k+\lambda+1,s)$. Take $v=2\tilde{U}$ in $A_j,\ 1\le j\le 5$. We obtain from Lemma 2, Lemma 6 and lemma 7 that for $r,\ s\ge 1$,

$$\begin{aligned}|A_1(\tilde{U})|\le\ & c(\tau^{1/2}\|U\|_{H^2(t,t+\tau;L^2_\omega(\Omega))}+(N^{-r}+h^{\tilde{s}})\|U_t\|_{M_\omega^{r+\frac14,\tilde{s}}(\Omega)}+\|(\vartheta-P_{N-1})U_t^*\|_\omega)\|\tilde{U}\|_\omega\\ \le\ & \|\tilde{U}\|_\omega^2+c(N^{-2r}+h^{2\tilde{s}})\|U\|^2_{C^1(0,T;M_\omega^{r+\frac14,\tilde{s}}(\Omega))}+c\tau\|U\|^2_{H^2(t,t+\tau;L^2_\omega(\Omega))},\end{aligned}$$

$$|A_2(\tilde{U})|\le c(N^{-r}+h^{\tilde{s}})(\|U\|^2_{Y_\omega^{r,0,\delta}(\Omega)}+(\|U\|_\infty+\|U^*\|_\infty)\|U\|_{M_\omega^{r+\frac14,\tilde{s}}(\Omega)})|\tilde{U}|_{1,\omega},$$

$$|A_3(\tilde{U})|\le c\nu\sigma\tau(\|U_t^*-U_t\|_{1,\omega}+\|U_t\|_{1,\omega})|\tilde{U}|_{1,\omega}\le c\nu\sigma\tau\|U\|_{C^1(0,T;H^1_\omega(\Omega))}|\tilde{U}|_{1,\omega},$$

$$|A_4(\tilde{U})|\le c(N^{-r}+h^{\bar{s}})\|P\|_{H^{\bar{s}}(I_y,H^1_\omega(I_x))\bigcap H^1(I_y,H_\omega^{r+1}(I_x))}(\|\tilde{U}\|_\omega+|\tilde{U}|_{1,\omega}),$$

$$|A_5(\tilde{U})|\le c(\|(\vartheta-P_c)f\|_\omega+\|(\vartheta-P_{N-1})f\|_\omega)\|\tilde{U}\|_\omega\le cN^{-r}\|f\|_{L^2(I_y,H^r_\omega(I_x))}\|\tilde{U}\|_\omega,$$

$$|A_6(w)|\le c(N^{-r}+h^{\bar{s}})\|P\|_{H^{\bar{s}}(I_y,H^2_\omega(I_x))\bigcap H^2(I_y,H_\omega^{r+1}(I_x))\bigcap H^1(I_y,H_\omega^{r+\frac54}(I_x))\bigcap L^2(I_y,H_\omega^{r+2}(I_x))}\|w\|_\omega,$$

$$|A_7(w)|\le c(N^{-r}+h^{\tilde{s}-1})((\|U^*\|_{1,\infty}+\|U\|_{1,\infty})\|U\|_{M_\omega^{r+1,\tilde{s}}(\Omega)}+\|U\|^2_{Y_{1,\omega}^{r,0,\delta}(\Omega)})\|w\|_\omega,$$

$$|A_8(w)|\le\ cN^{-r}\|f\|_{L^2(I_y,H_\omega^{r+1}(I_x))\bigcap H^1(I_y,H^r_\omega(I_x))}\|w\|_\omega.$$

Similarly, we take $v=m\tau\tilde{U}_t$ in $A_j(v)$, $1\le j\le 5$, and estimate them, such as

$$m\tau|A_1(\tilde{U}_t)|\le\varepsilon\tau\|\tilde{U}_t\|^2_\omega+\tfrac{cm^2\tau}{\varepsilon}(N^{-2r}+h^{2\tilde{s}})\|U\|^2_{C^1(0,T;M_\omega^{r+\frac14,\tilde{s}}(\Omega))}+\tfrac{cm^2\tau^2}{\varepsilon}\|U\|^2_{H^2(t,t+\tau;L^2_\omega(\Omega))}.$$

Moreover, by Lemma 5, Lemma 7 and Lemma 11,

$$\|\tilde{U}(0)\|^2_\omega+\tau|\tilde{U}(0)|^2_{1,\omega}\le c(1+2\tau N^4+c_d\tau h^{-2})(N^{-2r}+h^{2\tilde{s}})\|U_0\|^2_{M_\omega^{r+\frac14,\tilde{s}}(\Omega)},$$

$$\|U^*\|_{1,\infty}\le c\|U\|_{A_\omega^{\alpha,\beta}(\Omega)},\qquad\text{for }\alpha,\ \beta>\tfrac12.$$

Besides, if (5.5) holds and $r>\frac78,\ \bar{s}>\frac76,\ \tilde{s}>\frac{13}{6}$, then $\tau^2+N^{-2r}+h^{2\bar{s}}+h^{2(\tilde{s}-1)}=o(\frac{h}{N})$.

Finally, by an argument similar to the proof of Theorem 1, we have the following result.

Theorem 2. Assume that

(i) $\lambda\ge 1$, (4.1) and condition (i) of Theorem 1 hold;

(ii) for $r\ge 1,\ s>13/6$ and $\alpha,\ \beta>1/2$, $U\in C(0,T;M_\omega^{r+1,\tilde{s}}(\Omega)\bigcap W^{1,\infty}(\Omega)\bigcap A_\omega^{\alpha,\beta}(\Omega)\bigcap Y_{1,\omega}^{r,0,\delta}(\Omega))$ $\bigcap C^1(0,T;M_\omega^{r+\frac14,\tilde{s}}(\Omega))\bigcap H^2(0,T;L^2_\omega(\Omega))$;

(iii) for $r\ge 1$ and $s>\frac78$, $P\in C(0,T;H^{\bar{s}}(I_y,H^2_\omega(I_x))\bigcap H^2(I_y,H_\omega^{r+1}(I_x))\bigcap H^1(I_y,H_\omega^{r+2}(I_x)))$ and $f\in C(0,T;L^2(I_y,H_\omega^{r+1}(I_x))\bigcap H^1(I_y,H^r_\omega(I_x)))$.

Then there exists a positive constant d_3 depending only on ν and the norms of U and P in the spaces mentioned above such that for all $t \leq T$,

$$\|U(t) - u(t)\|_\omega \leq d_3(\tau + N^{-r} + h^{\bar{s}} + h^{\bar{s}-1}).$$

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