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Integral constants of Transformed geometric Poisson process [1]

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Abstract

In this paper, we introduce the conditions that the P-process has the intensity function which it is a standard form of gamma distribution. And we show that the transformed geometric Poisson process which the intensity function is a standard form of gamma distribution is a alternative sign P-process

Key Words and Phrases: P-process, transformed geometric Poisson process, alternative sign P-process.

1. Introduction

Park(1997a) introduced the P-process and transformed geometric Poisson process such that the intensity function is $g_i(t) \neq g_j(t)$ for $i \neq j$. And Park(1997 b) showed that the transformed geometric Poisson process which the intensity function is a Pareto distribution is a strongly P-process. In this paper, we will show that the transformed geometric Poisson process which the intensity function is a standard form of gamma distribution is a alternative sign P-process.

Let $\int_* f(t)dt = \int f(t)dt - C$, where C is a integral constant of $f(t)$.

Definition 1. The counting process $\{N(t)|t \geq 0\}$ is said to be a *polynomial process (P-process) with intensity function* $g_n(t)$ if

(i) $N(0) = 0$,

(ii) $P\{N(t+h) - N(t) = 1|N(t) = n\} = g_n(t)h + o(h)$

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where $-\infty < \left[\int_* g_n(t)dt\right]_{t=0} < \infty,$

(iii) $P\{N(t+h)-N(t) \geq 2|N(t)=n\} = o(h)$ for each $n = 0,\ 1,\ 2,\ \cdots$.

If $g_n(t) = \lambda$ for each $n = 0,\ 1,\ 2, \cdots$, then the P-process is a Poisson process with rate λ. And if $g_n(t) = \lambda(t)$ for each $n = 0,\ 1,\ 2,\ \cdots$, then the P-process is a nonhomogeneous Poisson process with intensity function $\lambda(t)$.

Let $P_n(t) = P\{N(t) = n\}$. Then, from the definition 1, we obtain that

$$P_0(t) = k_0 \exp\Big(-\int_* g_0(t)dt\Big)$$

and for $n \geq 1$,

$$P_n(t) = \exp\Big(-\int_* g_n(t)dt\Big)\Big[\int_* g_{n-1}(t)P_{n-1}(t)\exp\Big(\int g_n(t)dt\Big)dt\Big] + k_n \exp\Big(-\int_* g_n(t)dt\Big),$$

for some constants $k_0,\ k_1,\ \cdots$. The constants $k_0,\ k_1,\ k_2,\ \cdots$ is called to be a *integral constants* of P-process.

Let X be a geometric random variable. Then random variable $Y = X - 1$ is called to be a *transformed geometric.* Park(1997a) introduced a P-process which the distribution of number of events in interval $[0,t]$ is transformed geometric and $g_i(t) \neq g_j(t)$ for each $i \neq j$ $(i,j = 0,\ 1,\ 2,\ \cdots)$.

Definition 2. The P-process $\{N(t)|t \geq 0\}$ is said to be a *transformed geometric Poisson process* with intensity function $f(t)$ if

(i) $f(0) = 0$

(ii) $0 \leq f(t) < 1$ for each $t \geq 0$

(iii) $g_n(t) = (n+1)\frac{df(t)/dt}{1-f(t)}$

Definition 3. The P-process $\{N(t)|t \geq 0\}$ is called to be a *strongly P-process* if

$$k_0 = 1 \text{ and } k_n = 0 (n \geq 1).$$

Definition 4. The P-process $\{N(t)|t \geq 0\}$ is called to be a *alternative sign P-process* if

$$k_0 = 1 \text{ and } k_n = (-1)^n (n \geq 1).$$

Park(1997a, 1997b) showed that the transformed geometric Poisson process which intensity function is a Pareto distribution is a strongly P-process and (0,1)-generalized Poisson process is a strongly P-process but (1,2)-generalized Poisson process is not a strongly P-process. Also (1,2)-generalized Poisson process is not a alternative sign P-process.

II. Main results

In this section, we obtain the conditions that the P-process has the intensity function which it is a standard form of gamma distribution. And we show that the transformed geometric Poisson process which the intensity function is a standard form of gamma distribution is a alternative sign P-process

Theorem 1. If the counting process $\{N(t)|t \geq 0\}$ is satisfying

(1) $N(0) = 0$

(2) $P\{N(t+h) - N(t) = 1 | N(t) = k\} = \frac{(k+1)(\alpha-1-t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha)-t^{\alpha-1}e^{-t}} h + o(h)$

where $\alpha > 1$ and $k = 0, 1, 2, \cdots$

(3) $P\{N(t+h) - N(t) \geq 2 | N(t) = k\} = o(h) \quad (k = 0, 1, 2, \cdots)$,

then the counting process $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process which the intensity function is a standard form of gamma distribution.

Proof. Since

$$\begin{aligned} g_k(t) &= \frac{(k+1)(\alpha-1-t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha) - t^{\alpha-1}e^{-t}} \\ &= (k+1)\frac{\frac{d}{dt}(\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t})}{1 - (\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t})} \\ &= (k+1)\frac{\frac{d}{dt}f(t)}{1-f(t)} \end{aligned}$$

where $f(t) = \frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}$,

$$\begin{aligned}\int_* g_k(t)dt &= \int_*(k+1)\frac{\frac{d}{dtf(t)}}{1-f(t)}dt \\ &= -(k+1)\ln[1-f(t)] \\ &= -(k+1)\ln[1-\frac{1}{\Gamma(\alpha)}t^{\alpha-1}e^{-t}].\end{aligned}$$

Hence we obtain that

$$-\infty < \Big[\int_* g_n(t)dt\Big]_{t=0} < \infty.$$

Thus the counting process $\{N(t)|t \geq 0\}$ is a P-process.
And when $\alpha > 1$,

$$f(0) = 0$$

and

$$0 \leq f(t) < 1 \quad \text{for each} \quad t \geq 0.$$

Therefore, by Definition 2, the counting process $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process which the intensity function is a standard form of gamma distribution.

Theorem 2. If $\{N(t)|t \geq 0\}$ is a transformed geometric Poisson process which the intensity function is a standard form of gamma distribution. Then $\{N(t)|t \geq 0\}$ is a alternative sign P-process

Proof. Since

$$P_0(t) = k_0\exp\Big(-\int_* g_0(t)dt\Big)$$

and for $n \geq 1$

$$P_n(t) = \exp\Big(-\int_* g_n(t)dt\Big)\Big[\int_* g_n - 1(t)P_n - 1(t)\exp\Big(\int g_n(t)dt\Big)dt\Big] \\ +k_n\exp\Big(-\int_* g_n(t)dt\Big),$$

if $n = 0$,

$$g_0(t) = \frac{(\alpha - 1 - t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha) - t^{\alpha-1}e^{-t}}$$

and

$$\begin{aligned} P_0(t) &= k_0 \exp\Big(- \textstyle\int_* g_0(t)dt \Big) \\ &= k_0 (1 - \tfrac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t}). \end{aligned}$$

The boundary condition $P_0(0) = P\{N(0) = 0\} = 1$ implies that $k_0 = 1$.
Suppose $n \geq 1$. Since

$$g_n(t) = \frac{(n+1)(\alpha - 1 - t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha) - t^{\alpha-1}e^{-t}}$$

and

$$g_{n-1}(t) = \frac{n(\alpha - 1 - t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha) - t^{\alpha-1}e^{-t}},$$

We obtain

$$\begin{aligned} \int_* g_n(t)dt &= \int_* \frac{(n+1)(\alpha - 1 - t)t^{\alpha-2}e^{-t}}{\Gamma(\alpha) - t^{\alpha-1}e^{-t}} dt \\ &= -(n+1)\ln(1 - \frac{1}{\Gamma(\alpha)} t^{\alpha-1} e^{-t}). \end{aligned}$$

Since the intensity function is a standard form of gamma distribution, we know that

$$P_{n-1}(t) = (1 - \frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})(\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})^{n-1}$$

Thus,

$$\begin{aligned} P_n(t) &= \exp\Big(- \textstyle\int_* g_n(t)dt \Big) \Big[\textstyle\int_* g_{n-1}(t) P_{n-1}(t) \exp\Big(\textstyle\int_* g_n(t)dt \Big) dt \Big] \\ &\quad + k_n \exp\Big(- \textstyle\int_* g_n(t)dt \Big) \end{aligned}$$

$$= (1 - \frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})^{n+1} \int_* \frac{n \frac{((\alpha-1-x)x^{\alpha-2}e^{-x}}{\Gamma(\alpha)-x^{\alpha-1}e^{-x})} (\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})^{n-1}}{\{1-(\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})\}^n} dt$$
$$+k_n \{1 - (\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})\}^{n+1}$$
$$= \sum_{i=1}^{n} (-1)^{i+1} (\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})^{n-i} \{1 - (\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})\}^i$$
$$+k_n \{1 - (\frac{1}{\Gamma(\alpha)} t^{\alpha-2} e^{-t})\}^{n+1}.$$

The boundary condition $P_n(0) = 0$ implies that $k_n = (-1)^n$.

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