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Bayesian Procedure for Estimating Exponential Reliability Under the Censored Sample with Incomplete Information

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Abstract

This paper deals with the problem of obtaining some Bayes estimators of exponential reliability in a time censored sampling with incomplete information. Some Bayes estimators are proposed and studied under squared error loss and Harris loss.

Key Words and Phrases: Bayesian Estimation, Reliability, Incomplete, Censored Sample

1. Introduction

The exponential distribution plays an important role in many practical reliability analyses. It is the first model for which statistical methods were extensively developed and is widely used as a reliability model.

Elperin and Gertsbakh(1988) investigated the Bayes interval estimation for exponential parameter in a random censored sampling with incomplete information which includes, as particular cases, both the random censoring model and the quantal-response model. Calabria and Pulcini(1990) proposed the Bayesian procedure for estimating the exponential mean lifetime and the reliability function in a time censoring model with incomplete information using the squared error loss. Kim(1995) proposed the Bayesian procedure for estimating the Rayleigh reliability function under the censored sample with incomplete information.

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In this paper, we will study the Bayesian estimation for the reliability function, at a specified mission time t, based on a time censored sample with incomplete information observed from the exponential model. A noninformative prior, a locally uniform prior, and a beta prior distribution are considered as prior about a reliability. The squared error loss function and the Harris loss functions are considered.

2. Bayesian Estimation

We consider the Bayes estimators of the reliability, at a specified mission time t, for the exponential distribution, denoted by $E(\lambda)$, with probability density function(pdf) given as

$$f(x|\lambda) = \frac{1}{\lambda} exp\Big(- \frac{x}{\lambda}\Big), \quad 0 < x < \infty. \tag{1}$$

Then the reliability R at a specified time $t > 0$ is given by

$$R = P(X > t) = exp\Big(- \frac{t}{\lambda}\Big). \tag{2}$$

Let y be the fixed time to inspection and let x_i be the exponential lifetime of item $i(i = 1, 2, \ldots, n)$. Then three possibilities can be occurred in testing item i: First, the item fails at the instant $x_i(x_i < y)$. The failure is not signalled and the item is found failed on the inspection time y. In this case, the failure time is before the inspection time. Second, the item fails at the instant $x_i(x_i < y)$. The failure is immediately signalled. In this case, the failure time is exactly known. Third, the item is found unfailed at time y. In this case, the failure time would be beyond the inspection time.

When n items are tested, the corresponding likelihood function is given by

$$L(\lambda|\underline{x}) \propto \left(\frac{1}{\lambda}\right)^{n_2} e^{-\frac{u}{\lambda}}(1 - e^{-\frac{y}{\lambda}})^{n_1} e^{-\frac{y}{\lambda}n_3} \tag{3}$$

where n_1 is the number of elements in the set of noncensored and nonsignalled observations, n_2 is the number of elements in the set of noncensored and signalled observations, n_3 is the number of elements in the set of censored observations, and u is the sum of squares of the failure time of items in the set of noncensored and signalled observations. Then $n = n_1 + n_2 + n_3$ and the value of n_1 and n_2 depend on the value of the probability of failure-to-signal p. On the average $n_2/(n_1 + n_2) = p$.

Thus the likelihood function of R, which can be obtained from (3) by substituting $\lambda = -t/\ln R$, is

$$L(R|\underline{x}) \propto \Big(- \frac{1}{t} \ln R\Big)^{n_2} R^{\frac{u+yn_3}{t}} \Big(1 - R^{\frac{y}{t}}\Big)^{n_1} \tag{4}$$

Unfortunately, since the likelihood function is not exponential, a natural conjugate prior can not be found. Thus a prior density generally involves numerical integrations. But using the binomial formula, the likelihood function of R can be written as

$$L(R|\underline{x}) \propto \frac{1}{t^{n_2}} \sum_{j=0}^{n_1} c_j R^{\frac{w_j}{t}} (-\ln R)^{n_2}, \tag{5}$$

where $c_j = (-1)^j \binom{n_1}{j}$, $j = 0, 1, \ldots, n_1$ and $w_j = u + (n_3 + j)y$. Hence the commonly used informative prior for R yields some estimators of R which are in an analytical form.

Now, we consider the Jeffreys(1961)' noninformative(NI) prior for R. When the time t is specified, the noninformative prior for R is

$$\pi(R) = -\frac{1}{R \ln R}, \quad 0 < R < 1. \tag{6}$$

Then the product of (7) and (5) provides the posterior density of R:

$$\pi(R|\underline{x}) \propto \frac{1}{t^{n_2}} \sum_{j=0}^{n_1} c_j R^{\frac{w_j}{t}-1} (-\ln R)^{n_2-1}. \tag{7}$$

Since

$$\int_0^1 R^{s-1} (-\ln R)^{n-1} dR = \frac{\Gamma(n)}{s^n}, \tag{8}$$

the posterior density of R is given by

$$\pi(R|\underline{x}) = \frac{\sum_{j=0}^{n_1} c_j R^{\frac{w_j}{t}-1} (-\ln R)^{n_2-1}}{\Gamma(n_2) \sum_{j=0}^{n_1} c_j (\frac{t}{w_j})^{n_2}}, \quad 0 < R < 1, \tag{9}$$

where $\Gamma(n)$ is the gamma function defined by $\Gamma(n) = \int_0^\infty z^{n-1} e^{-z} dz$.

Thus the Bayes estimators can be calculated for a number of the loss functions.

If the squared error loss function $L_1(R, \hat{R}) = (R - \hat{R})^2$ is applied, then the Bayes estimator of the reliability function is given as follows:

Theorem 2.1. (Calabria and Pulcini(1990)) If the squared error loss function and a noninformative prior are used, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{NI,L_1} = \frac{T_1(w_j + t; n_2)}{T_1(w_j; n_2)}, \tag{10}$$

where

$$T_1(z_j;l) = \sum_{j=0}^{n_1} c_j z_j^{-l}. \tag{11}$$

Next, we consider a loss function suggested by Harris(1976) for the case $k = 2$, given by $L_2(R,\hat{R}) = [(1-R)^{-1} - (1-\hat{R})^{-1}]^2$. Under the Harris loss, the Bayes estimator $\hat{R}_{L_2}$ of R is $\hat{R}_{L_2} = 1 - [E\{(1-R)^{-1}|\underline{X} = \underline{x}\}]^{-1}$

Then one can obtain the Bayes estimator of the reliability function as follows:

Theorem 2.2. If the Harris loss function and a noninformative prior are used, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{NI,L_2} = \frac{T_2(w_j + mt + t; n_2)}{T_2(w_j + mt; n_2)}, \quad n_2 > 1, \tag{12}$$

where

$$T_2(z_{jm};l) = \sum_{j=0}^{n_1} c_j \sum_{m=0}^{\infty} z_{jm}^{-l}. \tag{13}$$

Proof. By transforming $Y = -\ln R$, one can obtain the following relation:

$$E\Big(\frac{1}{1-R}|\underline{X} = \underline{x}\Big) = \frac{\sum_{j=0}^{n_1} c_j \int_0^\infty y^{n_2-1} e^{-\frac{w_j}{t}y}(1-e^{-y})^{-1}dy}{\Gamma(n_2)\sum_{j=0}^{n_1} c_j (\frac{t}{w_j})^{n_2}}. \tag{14}$$

With the aid of the formula 4.5(10) in Erdélyi *et al.*(1955),

$$\int_0^\infty \frac{z^{\nu-1}e^{-pz}}{1-e^{-z/a}}dz = a^\nu \Gamma(\nu)\zeta(\nu, ap), \ Re(p) > 0, \ Re(\nu) > 1, \tag{15}$$

where $\zeta(s,\nu) = \sum_{m=0}^{\infty}(\nu+m)^{-s}$ $(Re(s) > 0)$ is the generalized zeta function, the Bayes estimator becomes

$$\hat{R}_{NI,L_2} = 1 - \frac{\sum_{j=0}^{n_1} c_j (\frac{t}{w_j})^{n_2}}{\sum_{j=0}^{n_1} c_j \zeta(n_2, \frac{w_j}{t})} = 1 - \frac{\sum_{j=0}^{n_1} c_j (\frac{t}{w_j})^{n_2}}{\sum_{j=0}^{n_1} c_j \sum_{m=0}^{\infty} (\frac{w_j}{t} + m)^{-n_2}}. \tag{16}$$

With the aid of the formula 1.10(2) in Erdélyi *et al.*(1953)

$$\zeta(s,\nu) = \zeta(s, 1+\nu) + \nu^{-s}, \tag{17}$$

the numerator of the Bayes estimator is

$$\sum_{j=0}^{n_1} c_j\zeta\left(n_2, \frac{w_j}{t}\right) - \sum_{j=0}^{n_1} c_j\left(\frac{t}{w_j}\right)^{n_2} = \sum_{j=0}^{n_1} c_j\zeta\left(n_2, 1+\frac{w_j}{t}\right) \tag{18}$$
$$= \sum_{j=0}^{n_1} c_j \sum_{m=0}^{\infty}\left(\frac{w_j+mt+t}{t}\right)^{-n_2}$$

Thus the Bayes estimator is

$$\hat{R}_{NI,L_2} = \frac{\sum_{j=0}^{n_1} c_j \sum_{m=0}^{\infty}(w_j+mt+t)^{-n_2}}{\sum_{j=0}^{n_1} c_j \sum_{m=0}^{\infty}(w_j+mt)^{-n_2}} \tag{19}$$

Next, we consider a locally uniform(LU) prior distribution $U^R(0,1)$ for R with pdf

$$\pi(R) = 1, \quad 0 < R < 1. \tag{20}$$

Then the posterior density of R is given by

$$\pi(R|\underline{x}) = \frac{\sum_{j=0}^{n_1} c_j R^{\frac{w_j}{t}}(-\ln R)^{n_2}}{\Gamma(n_2+1)\sum_{j=0}^{n_1} c_j(\frac{t}{w_j+t})^{n_2+1}}, \ 0 < R < 1, \tag{21}$$

where $\Gamma(n)$ is the gamma function.

With the squared error loss function, The Bayes estimator of the reliability is given as follows:

Theorem 2.3. If the squared error loss is used and R has a locally uniform prior, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{LU,L_1} = \frac{T_1(w_j+2t; n_2+1)}{T_1(w_j+t; n_2+1)}, \tag{22}$$

where $T_1(z_j; l)$ is given by the equation (11).

Proof. This can be easily proved from the equation (8), so we omit the proof.

Under the Harris loss one can obtain the Bayes estimator of the reliability function as follows:

Theorem 2.4. If the Harris loss function is used and R has a locally uniform prior, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{LU,L_2} = \frac{T_2(w_j+mt+2t; n_2+1)}{T_2(w_j+mt+t; n_2+1)}, \tag{23}$$

where $T_2(z_{jm}; l)$ is given by the equation (13).

Proof. The proof is similar to Theorem 2.2 and thus is omitted.

Since a locally uniform prior $U^R(0,1)$ does not depend upon R, the posterior density of R is proportional to the likelihood function of σ. Thus the generalized maximum likelihood estimator for a locally uniform prior for R is also the classical maximum likelihood estimator.

Finally, we consider a beta prior distribution $B^R(\alpha,\beta)$ for R with pdf

$$\pi(R) = \frac{1}{B(\alpha,\beta)} R^{\alpha-1}(1-R)^{\beta-1}, \quad 0 < R < 1,\ 0 < \alpha,\ \beta, \tag{24}$$

where $B(\alpha,\beta)$ is the beta function with parameters α and β defined by $B(\alpha,\beta) = \int_0^1 z^{\alpha-1}(1-z)^{\beta-1}dz$.

Then the product of (24) and (5) provides the posterior density function of R:

$$\pi(R|\underline{x}) \propto \frac{1}{B(\alpha,\beta)} \frac{1}{t^{n_2}} \sum_{j=0}^{n_1} c_j(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1}(1-R)^{\beta-1} \tag{25}$$

In order to the marginal distribution of X, the calculation of the integration $\int_o^1(-\ln R)^{n_2} R^{w_j/t+\alpha-1}(1-R)^{\beta-1}dR$ is needed. Expanding $(1-R)^{\beta-1}$ in a binomial series, we have

$$\begin{aligned} & \int_0^1(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1}(1-R)^{\beta-1}dR \\ = & \int_0^1(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1} \textstyle\sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k} R^k dR. \end{aligned} \tag{26}$$

In (26), it is understood that if the parameter β is a positive integer, the series terminates; that is, all terms with $k > \beta - 1$ are zero.

Since the interchange of integration and summation in (26) is justifiable, we can write (26) as

$$\begin{aligned} & \int_0^1(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1}(1-R)^{\beta-1}dR \\ = & \Gamma(n_2+1) \textstyle\sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}\left(\frac{t}{w_j+\alpha t+kt}\right)^{n_2+1} \end{aligned} \tag{27}$$

whenever $\alpha + w_j/t > 0$, $\beta + n_2 > 0$. Hence the posterior density function of R is given by

$$\pi(R|\underline{x}) = \frac{\sum_{j=0}^{n_1} c_j(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1}(1-R)^{\beta-1}}{\Gamma(n_2+1)\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}\left(\frac{t}{w_j+\alpha t+kt}\right)^{n_2+1}}, \quad 0 < R < 1. \tag{28}$$

or, equivalently,

$$\pi(R|\underline{x}) = \frac{\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}(-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha+k-1}}{\Gamma(n_2+1)\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty}(-1)^k\binom{\beta-1}{k}\left(\frac{t}{w_j+\alpha t+kt}\right)^{n_2+1}}, \quad 0 < R < 1. \tag{29}$$

If the squared error loss function is considered, then the Bayes estimator of the reliability function is given as follows:

Theorem 2.5. If the squared error loss function is used and R follows a beta prior distribution with parameters α and β, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{B(\alpha,\beta),L_1} = \frac{T_3(w_j + \alpha t + kt + t, \beta - 1; n_2 + 1)}{T_3(w_j + \alpha t + kt, \beta - 1; n_2 + 1)}, \ \beta + n_2 > 0, \tag{30}$$

where

$$T_3(z_{jk}, \gamma; l) = \sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty} (-1)^k \binom{\gamma}{k} z_{jk}^{-l}. \tag{31}$$

Proof. This can be easily proved, so we omit the proof.

Also with the Harris loss function, the following theorem can be obtained:

Theorem 2.6. If the Harris loss function is used and R follows a beta prior distribution with parameters α and β, then the Bayes estimator of the reliability function is given by

$$\hat{R}_{B(\alpha,\beta),L_2} = \frac{T_3(w_j + \alpha t + kt + t, \beta - 2; n_2 + 1)}{T_3(w_j + \alpha t + kt, \beta - 2; n_2 + 1)}, \ \beta + n_2 - 1 > 0, \tag{32}$$

where $T_3(z_{jk}, \gamma; l)$ is given by the equation (31).

Proof.

$$E\Big(\frac{1}{1-R}|\underline{X} = \underline{x}\Big) = \frac{\sum_{j=0}^{n_1} c_j \int_0^1 (-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1} (1-R)^{\beta-2} dR}{\Gamma(n_2+1) \sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} (\frac{t}{w_j+\alpha t+kt})^{n_2+1}}. \tag{33}$$

Since the integration of the above equation converges whenever $\beta + n_2 - 1 > 0$, by expanding $(1-R)^{\beta-2}$ in a binomial series we have

$$\begin{aligned} & \int_o^1 (-\ln R)^{n_2} R^{\frac{w_j}{t}+\alpha-1} (1-R)^{\beta-2} dR \\ = \ & \Gamma(n_2+1) \textstyle\sum_{k=0}^{\infty} (-1)^k \binom{\beta-2}{k} (\frac{t}{w_j+\alpha t+kt})^{n_2+1}. \end{aligned} \tag{34}$$

Then we can obtain the Bayes estimators as follows:

$$\hat{R}_{B(\alpha,\beta),L_2} = 1 - \frac{\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty} (-1)^k \binom{\beta-1}{k} (w_j + \alpha t + kt)^{-(n_2+1)}}{\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty} (-1)^k \binom{\beta-2}{k} (w_j + \alpha t + kt)^{-(n_2+1)}}. \tag{35}$$

By applying the Pascal triangle identity, we have the Bayes estimator of R

$$\hat{R}_{B(\alpha,\beta),L_2} = \frac{\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty}(-1)^k \binom{\beta-2}{k}(w_j+\alpha t+kt+t)^{-(n_2+1)}}{\sum_{j=0}^{n_1} c_j \sum_{k=0}^{\infty}(-1)^k \binom{\beta-2}{k}(w_j+\alpha t+kt)^{-(n_2+1)}}. \quad (36)$$

Comparing (30) and (32), we have $\hat{R}_{B(\alpha,\beta-1),L_1} = \hat{R}_{B(\alpha,\beta),L_2}$. Thus we have shown that the two families of Bayes estimators have the same form.

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