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On C.L.T. and L.I.L. for fuzzy random variables

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Abstract

In this paper we study central limit theorem(C.L.T.) and law of iterated logarithm (L.I.L.) for fuzzy random variables with respect to Hausdorff distance.

Key Words and Phrases: C.L.T., Fuzzy random variable, L.I.L.

1. Introduction

Since the concept of fuzzy random variables was introduced by Kwakernaak [7], many authors have studied law of large numbers (e.g., Miyakoshi and Shimbo [11], Stein and Talati [16], Puri and Ralescu15, Kruse 6 and Klement, Puri and Ralescu [5]). Recently, Hong and Kim [4] studied a law of large numbers with respect to the Hausdorff distance between the expected intervals of a fuzzy number. In this paper we investigate C.L.T. and L.I.L. with respect to the same distance as in Hong and Kim[4].

2. Preliminaries

We will now describe some concepts related to fuzzy random variables as they will be necessary for the discussion in the next sections. Throughout this paper, unless otherwise stated, we assume that a fuzzy number, $\tilde{a}$, is strictly normal, that is, there exists an element $x \in \mathcal{R}$ such that $h_{\tilde{a}}(x) = 1$, where $h_{\tilde{a}}$ is the membership function of $\tilde{a}$.

Let $\tilde{A}$ be a fuzzy number. We define two subsets as follow:

$$\begin{aligned} \tilde{A}_{\bar{\alpha}} &= \{x | h_{\tilde{A}} \geq \alpha\} \text{ for any } \alpha \in (0,1], \\ \tilde{A}_{\alpha} &= \{x | h_{\tilde{A}} > \alpha\} \text{ for any } \alpha \in [0,1), \end{aligned}$$

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where $h_{\tilde{A}}$ is the membership function of $\tilde{A}$.

Let $\tilde{P}(\mathcal{R})$ be the classes of all fuzzy numbers and $\tilde{P}_N(\mathcal{R})$ be the subclass of $\tilde{P}(\mathcal{R})$ satisfying the following three conditions :
(i) $\tilde{a} \in \tilde{P}(\mathcal{R})$
(ii) $h_{\tilde{a}}$ is upper semicontinuous and quasi-concave,
(iii) $supp\,\tilde{a} = cl(\tilde{a}_0)$ is compact, that is, the closure of $\{x|h_{\tilde{a}}(x) > 0\}$ is a compact subset of $\mathcal{R}$.

Let $\tilde{a}_1, \tilde{a}_2 \in \tilde{P}_N(\mathcal{R})$ and $\tilde{a}_1 + \tilde{a}_2$ be the sum of $\tilde{a}_1$ and $\tilde{a}_2$. Then, according to Nguyen[14], we have

$$\begin{aligned}
&(\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}} = (\tilde{a}_1)_{\tilde{\alpha}} + (\tilde{a}_2)_{\tilde{\alpha}},\\
&\sup(\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}} = \bar{a}_{\tilde{\alpha}}^{(1)} + \bar{a}_{\tilde{\alpha}}^{(2)},\\
&\inf(\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}} = \underline{a}_{\tilde{\alpha}}^{(1)} + \underline{a}_{\tilde{\alpha}}^{(2)},\\
&\sup(\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}},\ \inf(\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}} \in (\tilde{a}_1 + \tilde{a}_2)_{\tilde{\alpha}} \text{ for any } \alpha \in (0,1],
\end{aligned}$$

where $\bar{a}_{\tilde{\alpha}}^{(k)} = \sup(\tilde{a}_k)_{\tilde{\alpha}}$ and $\underline{a}_{\tilde{\alpha}}^{(k)} = \inf(\tilde{a}_k)_{\tilde{\alpha}}$ $(k = 1, 2)$, and sup, inf denote the supremum and infimum, respectively.

Let $\tilde{A} \in \tilde{P}_N(\mathcal{R})$. The expected interval of a fuzzy number $\tilde{A}$ is denoted by $\int_0^1 \tilde{A}_{\tilde{\alpha}} d\alpha$ and defined as $\int_0^1 \tilde{A}_{\tilde{\alpha}} d\alpha = [\int_0^1 \bar{A}_{\tilde{\alpha}} d\alpha, \int_0^1 \underline{A}_{\tilde{\alpha}} d\alpha]$(see [3, Lemma 3]).

We define a distance between fuzzy numbers in $\tilde{P}_N(\mathcal{R})$ by

$$d(\tilde{A}, \tilde{B}) = d(\int_0^1 \tilde{A}_{\tilde{\alpha}} d\alpha, \int_0^1 \tilde{B}_{\tilde{\alpha}} d\alpha).$$

Here d denotes the Hausdorff distance between two compact subsets of $\mathcal{R}$. More precisely,

$$d(M, N) = \max\{\sup_{a \in M} \inf_{b \in N} |a - b|, \sup_{b \in N} \inf_{a \in M} |a - b|\}.$$

We review the definition of a fuzzy random variable and the one of the expectation of a fuzzy random variable. For details we refer to Kwakernaak[6].

Let $(\Omega, \mathcal{A}, P)$ be a probability space and the ω stand for the element of Ω.

Definition 1 *A fuzzy random variable ξ is defined as a function from Ω to $\tilde{P}_N(\mathcal{R})$ satisfying the following two conditions:*

(i) $\bar{X}_{\tilde{\alpha}}(\omega)$, $\underline{X}_{\tilde{\alpha}}(\omega) \in \xi(\omega)_{\tilde{\alpha}}$ for any $\alpha \in (0, 1]$,

(ii) $\bar{X}_{\tilde{\alpha}}$ and $\underline{X}_{\tilde{\alpha}}$ are $\mathcal{A}$-measurable for any $\alpha \in (0, 1]$, where $\bar{X}_{\tilde{\alpha}}(\omega) = \sup(\xi(\omega))_{\tilde{\alpha}}$ and $\underline{X}_{\tilde{\alpha}}(\omega) = \inf(\xi(\omega))_{\tilde{\alpha}}$.

Definition 2 *The expectation of a fuzzy random variable* ξ, *denoted by* $E\xi$, *is defined as a fuzzy numbers whose* α*-level set* $(E\xi)_{\bar{\alpha}}$ *is given by a closed interval* $[E\underline{X}_{\bar{\alpha}}, E\bar{X}_{\bar{\alpha}}]$ *for each* α, *where the symbol* E *in the square brackets stands for the expectation of random variables* $\underline{X}_{\bar{\alpha}}$ *and* $\bar{X}_{\bar{\alpha}}$ *with respect to* P*-measure.*

Next we give a definition of independence of a sequence of fuzzy random variables.

Definition 3 *Let* $\sigma(\bar{X}_{\bar{\alpha}}^{(n)}, \underline{X}_{\bar{\alpha}}^{(n)}; \alpha \in (0,1])$ *be the smallest* σ*-fields such that* $\bar{X}_{\bar{\alpha}}^{(n)}$ *and* $\underline{X}_{\bar{\alpha}}^{(n)}$ *for any* $\alpha \in (0,1]$ *are measurable. A sequence of fuzzy random variables* $\{\xi_n, n \geq 1\}$ *is called independent and identically distributed(i.i.d) if all finite subsequences of a sequence of* σ*-fields* $\{\sigma(\bar{X}_{\bar{\alpha}}^{(n)}, \underline{X}_{\bar{\alpha}}^{(n)}; \alpha \in (0,1]), n \geq 1\}$ *are mutually independent and* $\{(\bar{X}_{\bar{\alpha}}^{(n)}, \underline{X}_{\bar{\alpha}}^{(n)}), n \geq 1\}$ *are identically distributed for each* $\alpha > 0$.

If $\xi : \Omega \to \tilde{P}_N(\mathcal{R})$, then $supp\,\xi$ is the random interval which associates to $\omega \in \Omega$ the support of the fuzzy number $\xi(\omega)$. We define the norm $\|M\|$ of a compact set M in $\mathcal{R}$ as $\|M\| = \sup_{a \in M} |a|$.

3. Main Results

Theorem 1 (C.L.T.) *Let* $\{\xi_k, k \geq 1\}$ *be i.i.d fuzzy random variables with* $E\|supp\,\xi_1\|^2 < \infty$. *Then we have for every* $t \in \mathcal{R}$

$$\begin{aligned}
&\lim_{n\to\infty} P\{n^{-\frac{1}{2}} d(\sum_{k=1}^{n} \xi_k, \sum_{k=1}^{n} E\xi_k) \leq t\} \\
\leq\ & 2\Phi(t \cdot [\min\{E(\int_0^1 (\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha)^2,\ E(\int_0^1 (\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha)^2\}]^{-1}) - 1,
\end{aligned}$$

where Φ *is the standard normal distribution.*

Proof We first note that

$$\begin{aligned}
& P\{n^{-\frac{1}{2}} d(\sum_{k=1}^{n} \xi_k, \sum_{k=1}^{n} E\xi_k) \leq t\} \\
=\ & P\{n^{-\frac{1}{2}} \max\{|\int_0^1 \sum_{k=1}^{n} (\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha|,\ |\int_0^1 \sum_{k=1}^{n} (\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha|\} \leq t\} \\
\leq\ & \min\{\ P\{|n^{-\frac{1}{2}} \sum_{k=1}^{n} \int_0^1 (\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha| \leq t\}, \\
& \qquad P\{|n^{-\frac{1}{2}} \sum_{k=1}^{n} \int_0^1 (\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha| \leq t\}\ \}.
\end{aligned}$$

Since $\{\int_0^1(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha, k \geq 1\}$ are i.i.d random variables with mean 0 and variance $E[\int_0^1(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha]^2$, we have, as $n \to \infty$,

$$P\{|n^{-\frac{1}{2}}\sum_{k=1}^{n}\int_0^1(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha| \leq t\} \longrightarrow 2\Phi(t \cdot [E(\int_0^1(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha)^2]^{-1}) - 1.$$

In the similar manner we have, as $n \to \infty$,

$$P\{|n^{-\frac{1}{2}}\sum_{k=1}^{n}\int_0^1(\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha| \leq t\} \longrightarrow 2\Phi(t \cdot [E(\int_0^1(\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha)^2]^{-1}) - 1.$$

Therefore, we have the desired result.

Theorem 2 (L.I.L.) *Let $\{\xi_k, k \geq 1\}$ be i.i.d fuzzy random variables with $E\|supp\,\xi_1\|^2 < \infty$. Then we have*

$$\begin{aligned}
&\limsup(2n\log\log n)^{-1}d(\sum_{k=1}^{n}\xi_k, \sum_{k=1}^{n}E\xi_k) \\
\leq\ &\max\{[E(\int_0^1(\bar{X}_{\bar{\alpha}}^{(1)} - E\bar{X}_{\bar{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}}, [E(\int_0^1(\underline{X}_{\bar{\alpha}}^{(1)} - E\underline{X}_{\bar{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}}\} \quad a.s.
\end{aligned}$$

Proof

$$\begin{aligned}
&\limsup(2n\log\log n)^{-\frac{1}{2}}d(\sum_{k=1}^{n}\xi_k, \sum_{k=1}^{n}E\xi_k) \\
=\ &\limsup(2n\log\log n)^{-\frac{1}{2}}\max\{|\int_0^1\sum_{k=1}^{n}(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha|, \\
&\qquad |\int_0^1\sum_{k=1}^{n}(\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha|\} \\
=\ &\max\{\limsup(2n\log\log n)^{-\frac{1}{2}}|\int_0^1\sum_{k=1}^{n}(\bar{X}_{\bar{\alpha}}^{(k)} - E\bar{X}_{\bar{\alpha}}^{(k)})d\alpha|, \\
&\qquad \limsup(2n\log\log n)^{-\frac{1}{2}}|\int_0^1\sum_{k=1}^{n}(\underline{X}_{\bar{\alpha}}^{(k)} - E\underline{X}_{\bar{\alpha}}^{(k)})d\alpha|\} \\
\leq\ &\max\{[E(\int_0^1(\bar{X}_{\bar{\alpha}}^{(1)} - E\bar{X}_{\bar{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}}, [E(\int_0^1(\underline{X}_{\bar{\alpha}}^{(1)} - E\underline{X}_{\bar{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}}\}
\end{aligned}$$

Corollary 1 *If* $P\{\xi_1(\omega) \in$ a set of same type of symmetric fuzzy numbers$\} = 1$, *then under the conditions of Theorem 1, we have*

$$\lim_{n\to\infty} P\{n^{-\frac{1}{2}} d(\sum_{k=1}^{n} \xi_k, \sum_{k=1}^{n} E\xi_k) \le t\} \le 2\Phi(t \cdot [\min\{E(\int_0^1 (\bar{X}_{\tilde{\alpha}}^{(1)} - E\bar{X}_{\tilde{\alpha}}^{(1)})d\alpha)^2]^{-1}) - 1.$$

Proof If $P\{\xi_1(\omega) \in$ a set of same type of symmetric fuzzy numbers$\} = 1$, then we can easily check that

$$|\int_0^1 \sum_{k=1}^{n} (\bar{X}_{\tilde{\alpha}}^{(k)} - E\bar{X}_{\tilde{\alpha}}^{(k)})d\alpha| = |\int_0^1 \sum_{k=1}^{n} (\underline{X}_{\tilde{\alpha}}^{(k)} - E\underline{X}_{\tilde{\alpha}}^{(k)})d\alpha| \ a.s.$$

for all n. Now, following the line of the proof of Theorem 1 we can get the result.

Similarly, we have the following corollary.

Corollary 2 *If* $P\{\xi_1(\omega) \in$ a set of same type of symmetric fuzzy numbers$\} = 1$, *then under the conditions of Theorem 2, we have*

$$\limsup_{n\to\infty} (2n \log\log n)^{-1} d(\sum_{k=1}^{n} \xi_k, \sum_{k=1}^{n} E\xi_k) = [E(\int_0^1 (\bar{X}_{\tilde{\alpha}}^{(1)} - E\bar{X}_{\tilde{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}} \ a.s.$$

Example 1 Let ξ_n be i.i.d with $P\{\xi_1 = \tilde{a}_1\} = P\{\xi_1 = \tilde{a}_2\} = \frac{1}{2}$, where $\tilde{a}_1$ and $\tilde{a}_2$ are symmetric triangular fuzzy numbers with centers 0, 2 and spreads 1, 2, respectively. Then $P\{\underline{X}_{\tilde{\alpha}}^{(1)} = 1 - \alpha\} = P\{\underline{X}_{\tilde{\alpha}}^{(1)} = 2\alpha\} = \frac{1}{2}$, and hence we can easily compute that $[E(\int_0^1 (\underline{X}_{\tilde{\alpha}}^{(1)} - E\underline{X}_{\tilde{\alpha}}^{(1)})d\alpha)^2]^{\frac{1}{2}} = \frac{1}{4}$. Therefore, we have $\limsup_{n\to\infty} (2n \log\log n)^{-1} d(\sum_{k=1}^{n} \xi_k, \sum_{k=1}^{n} E\xi_k) = \frac{1}{4}$.

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