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APPROXIMATIONS FOR JUMP-DIFFUSION PROCESSES WITH NON-LIPSCHITZIAN COEFFICIENTS

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ABSTRACT. In this work, we consider Euler approximation for one-dimensional jump-diffusion processes under Yamada-Watanabe type conditions on the coefficients which turns out to converge to the strong solution in uniform L^1-sense.

1. Introduction

Let $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t\geq 0})$ be a complete probability space with $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. We consider an one-dimensional stochastic differential equation with Poisson jumps:

(1)

$$X_t = X_0 + \int_0^t \mu(X_{s-})\, ds + \int_0^t \sigma(X_{s-})\, dB_s + \int_0^t \int c(X_{s-}, u)\tilde{\nu}(ds, du)$$

where $\mu(x)$ and $c(x,u)$ are $\Re$-valued and $\sigma(x) = (\sigma_1(x), \cdots, \sigma_d(x))$ is $\Re^d$-valued for $x, u \in \Re$. $\{B_t, \mathcal{F}_t\}$ is a standard d-dimensional Brownian motion,

$$\begin{aligned} \tilde{\nu}(ds, dy) &:= \nu(ds, dy) - E\,\nu(ds, dy) \\ &:= \nu(ds, dy) - ds\,\Pi(dy) \end{aligned}$$

is a compensated Poisson random measure on $[0, \infty) \times \Re$ for some σ-finite measure Π, and we assume that $\{B_t\}$ and $\{\nu(dt, dy)\}$ are independent.

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It is well known that there exists a unique strong solution of equation (1) under the Lipschitz type conditions and linear growth conditions on the coefficients by using the Picard iteration method. (Gihman & Skorohod [1]) For a continuous Ito diffusion (i.e. $c \equiv 0$), Yamada and Watanabe [5] weakened the Lipschitz conditions on μ and σ and showed the pathwise uniqueness of the solution which implies the existence of unique strong solution. Furthermore, Kaneko and Nakao [2] showed that Euler approximation converges to the strong solution in the uniform L^2-sense provided the suitable conditions for existence and pathwise uniqueness of solutions of (1) hold. For a diffusion with jumps (i.e. $c \not\equiv 0$), Situ [4] dealt with strong solutions of (1) with $\Pi(dz) = dz/z^2$ and possibly discontinuous drift coefficient μ. In fact, he imposed the uniform ellipticity and a continuity condition on σ weaker than the Lipschitz type, and linear growth conditions on the coefficients. More recently, Mao [3] established the existence and uniqueness of strong solution for stochastic differential equation driven by continuous spatial semimartingale. He derived the solution by showing that the Euler approximation converges to the solution in the uniform L^1-sense, where continuity of the coefficients is assumed to be weaker than Lipschitz type. The aim of this work is to obtain existence of unique strong solution of (1) under Yamada-Watanabe type conditions of the coefficients via Euler approximation which turns out to converge to the solution in uniform L^1-sense. Compared to Situ's result [4], we deal with more general compensator measure Π, but need continuity of the drift coefficient. We find the technique in Mao [3] quite useful in our setting but extra work is required to handle the jump terms.

Throughout this work, we denote a positive generic constant by C, whose value differs from line to line.

2. Main results

We will consider the following assumptions on the coefficients of (1) to obtain the main result.

(A1) X_0 is an $\mathcal{F}_0$-measurable random variable such that $E|X_0|^2 < \infty$.

(A2) There exists a constant C such that for all $x \in \Re$

$$\int |c(x,u)|^2 \, \Pi(du) \;\le\; C(1+|x|^2).$$

(A3) For all $x, y \in \Re$,

$$|\mu(x) - \mu(y)| \leq \kappa_1(|x-y|),$$
$$\int |c(x,u) - c(y,u)|\, \Pi(du) \leq \kappa_2(|x-y|),$$

where $\kappa_i : \Re_+ \to \Re_+$ is a continuous nondecreasing concave function such that $\kappa_i(0) = 0$, $\int_{0+} \frac{du}{\kappa_i(u)} = \infty$, $i = 1, 2$.

(A4) For all $x, y \in \Re$,

$$\|\sigma(x) - \sigma(y)\|^2 \leq \rho(|x-y|^{2\alpha})$$

where $1/2 \leq \alpha \leq 1$, $\rho : \Re_+ \to \Re_+$ is a continuous nondecreasing concave function such that $\rho(0) = 0$, $\int_{0+} \frac{du}{\rho(u^{2\alpha})} = \infty$, and moreover, if $1/2 < \alpha \leq 1$ then there exists another continuous nondecreasing concave function $\eta : \Re_+ \to \Re_+$ such that $\eta(0) = 0$, $u^{-1}\rho(u^{2\alpha}) \leq \eta(u)$.

(A5) For any $N > 0$,

$$\lim_{\epsilon \downarrow 0} \sup_{|x| \leq N} \int_{|u| \leq \epsilon} |c(x,u)|^2\, \Pi(du) = 0.$$

Now we introduce the Euler approximation for (1). For each n, we define $\{X_t^n\}$ as follows:

$$(2) \qquad X_0^n = X_0,$$
$$X_t^n = X_{\frac{i}{n}}^n + \mu(X_{\frac{i}{n}-}^n)(t - i/n) + \sum_{j=1}^{d} \sigma_j(X_{\frac{i}{n}-}^n)(B_j(t) - B_j(i/n))$$
$$+ \int_{\frac{i}{n}}^{t} \int c(X_{\frac{i}{n}-}^n, u)\, \tilde{\nu}(ds, du)$$
$$\text{if } i/n < t \leq (i+1)/n, \quad i = 0, 1, 2, \cdots.$$

We denote by $D[0,T]$ the space of the functions which are right continuous and having left-hand limits on $[0,T]$.

THEOREM. *Let* (A1)-(A5) *hold. Then there exists a unique solution* $\{X_t\}$ *of equation* (1) *whose sample paths are in* $D[0,T]$. *Moreover*

$$\lim_{n\to\infty} E\left(\sup_{0 \leq t \leq T} |X_t - X_t^n| \right) = 0,$$

and

$$E\left(\sup_{0\le t\le T}|X_t|^2\right) < \infty.$$

Before we prove our main result, we need to present preliminary lemmas. We introduce simple processes

$$\bar{X}_t^n = \sum_{i=0}^{\infty} X^n_{\frac{i}{n}-}\chi_{[i/n,(i+1)/n)}(t) \quad \text{for} \quad n = 1, 2, \cdots .$$

Then (2) can be written as

(3)

$$X_t^n = X_0 + \int_0^t \mu(\bar{X}^n_{s-})\,ds + \int_0^t \sigma(\bar{X}^n_{s-})\,dB_s + \int_0^t\int c(\bar{X}^n_{s-},u)\,\tilde{\nu}(ds,du).$$

LEMMA 1. *Assume that* (A1) *holds and there exists a constant* $C>0$ *such that for all* $x\in\Re$,

$$(4)\qquad |\mu(x)|^2 + |\sigma(x)|^2 + \int |c(x,u)|^2\,\Pi(du) \le C(1+|x|^2).$$

Then for any $T>0$,

$$(5)\qquad E\left(\sup_{0\le t\le T}|X_t^n|^2\right) \le C_1 e^{C_2 T},$$

and

$$(6)\qquad \sup_{0\le t\le T} E|X_t^n - \bar{X}_t^n|^2 \le C_3/n,$$

where C_1, C_2 *and* C_3 *are independent of* n.

Proof. It follows from (3) that for $t\le T$,

$$E\left(\sup_{0\le s\le t}|X_s^n|^2\right) \le 4\,E|X_0|^2 + 4\,E\left(\int_0^t |\mu(\bar{X}^n_{s-})|\,ds\right)^2$$
$$+4\,E\left(\sup_{0\le s\le t}\left|\int_0^s \sigma(\bar{X}^n_{\tau-})\,dB_\tau\right|^2\right) + 4E\left(\sup_{0\le s\le t}\left|\int_0^s\int c(\bar{X}^n_{\tau-},u)\tilde{\nu}(d\tau,du)\right|^2\right)$$

Using (4), Hölder's inequality and Doob's martingale inequality we have

$$(7)\qquad E\left(\sup_{0\le s\le t}|X_s^n|^2\right) \le C_1 + C_2\int_0^t E\left(\sup_{0\le u\le s}|X_u^n|^2\right) ds$$

for some positive constants C_1 and C_2 which are independent of n. By the Gronwall's inequality we obtain (5). Similarly to (7), (6) follows easily. □

LEMMA 2. *Let* (A1)-(A4) *hold. Then for any* $T > 0$

$$(8)\qquad \sup_{0\le t\le T} E|X_t^m - X_t^n| \longrightarrow 0 \quad \text{as } m,n\to\infty.$$

Proof. The proof runs similarly to the proof of Lemma 2.4 of [3], even for the jump term. We only need an additional fact that $\kappa(u) = \kappa_1(u) + 2\,\kappa_2(u)$ is still concave with $\int_{0+}\frac{du}{\kappa(u)} = \infty$, which was derived in Lemma 1 of [4]. □

Now we are ready to provide the proof of main result.

Proof of Theorem. For $m > n \ge 1$,

$$(9)\qquad E\left(\sup_{0\le t\le T}|X_t^m - X_t^n|\right) \le E\left(\sup_{0\le t\le T}\left|\int_0^t (\mu(\bar X_{s-}^m) - \mu(\bar X_{s-}^n))\,ds\right|\right)$$
$$+E\left(\sup_{0\le t\le T}\left|\int_0^t (\sigma(\bar X_{s-}^m) - \sigma(\bar X_{s-}^n))\,dB_s\right|\right)$$
$$+E\left(\sup_{0\le t\le T}\left|\int_0^t\!\!\int (c(\bar X_{s-}^m,u) - c(\bar X_{s-}^n,u))\,\tilde\nu(ds,du)\right|\right).$$

By (A3) and Jensen's inequality, we have

$$(10)\quad E\left(\sup_{0\le t\le T}\left|\int_0^t (\mu(\bar X_{s-}^m) - \mu(\bar X_{s-}^n))\,ds\right|\right) \le \int_0^T \kappa_1(E|\bar X_s^m - \bar X_s^n|)\,ds.$$

If (A4) holds with $1/2 < \alpha \leq 1$, then by Burkholder-Davis-Gundy inequality we have

$$(11)\quad E\left(\sup_{0\leq t\leq T}\left|\int_0^t (\sigma(\bar{X}^m_{s-}) - \sigma(\bar{X}^n_{s-}))\, dB_s\right|\right)$$
$$\leq C\sum_{j=1}^d E\left\{\int_0^T |\sigma_j(\bar{X}^m_s) - \sigma_j(\bar{X}^n_s)|^2 ds\right\}^{1/2}$$
$$\leq C\left\{\int_0^T \rho(E|\bar{X}^m_s - X^m_s|^{2\alpha})ds\right\}^{1/2} + C\left\{\int_0^T \rho(E|X^n_s - \bar{X}^n_s|^{2\alpha})ds\right\}^{1/2}$$
$$+\frac{1}{2}E(\sup_{0\leq t\leq T}|X^m_s - X^n_s|) + C\int_0^T \eta(E|X^m_s - X^n_s|)\, ds,$$

where C is independent of m and n. If (A4) holds with $\alpha = 1/2$, it is even easier. Again by Burkholder-Davis-Gundy inequality we have that for any $\epsilon > 0$,

$$(12)\quad E\left(\sup_{0\leq t\leq T}\left|\int_0^t\int (c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u))\, \tilde{\nu}(ds, du)\right|\right)$$
$$\leq C\, E\left\{\int_0^T\int_{|u|>\epsilon} (c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u))^2\, \nu(ds, du)\right\}^{1/2}$$
$$+C\left\{E\int_0^T\int_{|u|\leq\epsilon} (c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u))^2\, \Pi(du)ds\right\}^{1/2},$$

where C is independent of m, n and ϵ. Let

$$Y_t = \int_0^t\int_{|u|>\epsilon} u\, \nu(ds, du).$$

Then there exist a random number σ and random points $\tau_1, \tau_2, \cdots, \tau_\sigma$ on $[0, T]$ such that

$$
\begin{aligned}
(13) \qquad & E\left\{\int_0^T \int_{|u|>\epsilon} |c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u)|^2 \, \nu(ds, du)\right\}^{1/2} \\
&= E\left\{\sum_{k=1}^{\sigma} |c(\bar{X}^m_{\tau_{k-}}, \Delta Y_{\tau_k}) - c(\bar{X}^n_{\tau_{k-}}, \Delta Y_{\tau_k})|^2\right\}^{1/2} \\
&\le E\sum_{k=1}^{\sigma} |c(\bar{X}^m_{\tau_{k-}}, \Delta Y_{\tau_k}) - c(\bar{X}^n_{\tau_{k-}}, \Delta Y_{\tau_k})| \\
&= \int_0^T \int_{|u|>\epsilon} E|c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u)| \, \Pi(du)ds \\
&\le \int_0^T \kappa_2(E|\bar{X}^m_s - \bar{X}^n_s|) \, ds.
\end{aligned}
$$

By (5) we get that for any n, $\epsilon > 0$ and $N > 0$,

$$
\begin{aligned}
& P\left(\left|\int_0^T \int_{|u|\le\epsilon}\right| c(\bar{X}^n_{s-}, u)|^2 \Pi(du)ds| > \delta\right) \\
&\le P(\sup_{0\le s\le T} |\bar{X}^n_s| > N) + \frac{1}{\delta} E \int_0^T \int_{|u|\le\epsilon} |c(\bar{X}^n_{s-}, u)|^2 \, \chi_{\{|\bar{X}^n_{s-}|\le N\}} \, \Pi(du)ds \\
&\le \frac{C_1}{N^2} e^{C_2 T} + \frac{T}{\delta} \sup_{|x|\le N} \int_{|u|\le\epsilon} |c(x, u)|^2 \Pi(du).
\end{aligned}
$$

By (A5) this implies that for any n,

$$
\int_0^T \int_{|u|\le\epsilon} |c(\bar{X}^n_{s-}, u)|^2 \, \Pi(du)ds
$$

converges to zero in probability as $\epsilon \to 0$. From (12) and (13) we have

$$
\begin{aligned}
(14) \qquad & E\left(\sup_{0\le t\le T} \left| \int_0^t \int (c(\bar{X}^m_{s-}, u) - c(\bar{X}^n_{s-}, u)) \, \tilde{\nu}(ds, du)\right|\right) \\
&\le C \int_0^T \kappa_2(E|\bar{X}^m_s - \bar{X}^n_s|) \, ds,
\end{aligned}
$$

where C is independent of m and n. Combining (9), (10), (11) and (14) we obtain that

$$\begin{aligned} &E\left(\sup_{0\le t\le T}|X_t^m - X_t^n|\right) \\ &\le 2\int_0^T \kappa_1(E|\bar{X}_s^m - \bar{X}_s^n|)\,ds + C\left\{\int_0^T \rho(E|\bar{X}_s^m - \bar{X}_s^n|)\,ds\right\}^{1/2} \\ &+C\left\{\int_0^T \rho(E|\bar{X}_s^m - X_s^m|^{2\alpha})ds\right\}^{1/2} + C\left\{\int_0^T \rho(E|X_s^n - \bar{X}_s^n|^{2\alpha})ds\right\}^{1/2} \\ &+C\int_0^T \eta(E|X_s^m - X_s^n|)\,ds + C\int_0^T \kappa_2(E|\bar{X}_s^m - \bar{X}_s^n|)\,ds, \end{aligned}$$

from which by (6) and (8),

$$E\left(\sup_{0\le t\le T}|X_t^m - X_t^n|\right) \longrightarrow 0 \quad \text{as } m,n\to\infty.$$

By routine arguments, it is not hard to show that there exists $\{X_t\}$ whose sample paths are in $D[0,T]$ such that

$$\lim_{n\to\infty} E\left(\sup_{0\le t\le T}|X_t - X_t^n|\right) = 0,$$

and it is the unique solution of equation (1) satisfying

$$E\left(\sup_{0\le t\le T}|X_t|^2\right) < \infty. \qquad \square$$

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