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REAL HYPERSURFACES SATISFYING $\nabla_\xi S = 0$ OF A COMPLEX SPACE FORM

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ABSTRACT. The main purpose of this paper is to prove that if a real hypersurface M of a complex space form satisfies $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ for some constant on σ on M, then the structure vector field ξ is principal, where S denotes the Ricci tensor of M.

1. Introduction

An n-dimensional complex space form $M^n(c)$ is a Kaehlerian manifold of constant holomorphic sectional curvature c. A complete and simply connected complex space forms are isometric to a complex projective space CP^n, a complex Euclidean space E^n or a complex hyperbolic space CH^n according as $c > 0$, $c = 0$ or $c < 0$.

Let M be a real hypersurface of $M^n(c)$, $c \neq 0$. Then M has an almost contact metric structure (ϕ, ξ, η, g) induced from the Kaehlerian metric and complex structure J of $M^n(c)$. The structure vector ξ is said to be *principal* if $A\xi = \alpha\xi$, where A is the shape operator in the direction of the unit normal C and $\alpha = \eta(A\xi)$. We denote by ∇ and S, the Levi-Civita connection with respect to the Riemannian metric tensor g and the Ricci tensor of type (1,1) on M respectively. There exist many studies about real hypersurfaces of $M^n(c)$. One of the first studies is the classification of homogeneous real hypersurfaces of a complex projective space CP^n by Takagi ([9]), who showed that these hypersurfaces of CP^n could be divided into six types which are said to

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be of type A_1, A_2, B, C, D and E, and in ([3]) Cecil-Ryan and Kimura ([6]) proved that they are realized as the tubes of constant radius over Kaehlerian submanifolds.

THEOREM A ([6]). *Let M be a connected real hypersurface of CP^n. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following;*

- (A_1) *a geodesic hypersphere (that is, a tube of radius r over a hyperplane CP^{n-1}), where $0 < r < \frac{\pi}{2}$,*
- (A_2) *a tube of radius r over a totally geodesic CP^k $(1 \leq k \leq n-2)$, where $0 < r < \frac{\pi}{2}$,*
- (B) *a tube of radius r over a complex quadric Q^{n-1}, where $0 < r < \frac{\pi}{4}$,*
- (C) *a tube of radius r over $CP^1 \times CP^{\frac{n-1}{2}}$, where $0 < r < \frac{\pi}{4}$ and $n(\geq 5)$ is odd,*
- (D) *a tube of radius r over a complex Grassmann $G_{2,5}(C)$, where $0 < r < \frac{\pi}{4}$ and $n = 9$,*
- (E) *a tube of radius r over a Hermitian symmetric space $SO(10)/U(5)$, where $0 < r < \frac{\pi}{4}$ and $n = 15$.*

Recently Berndt ([2]) showed that all real hypersurfaces of a complex hyperbolic space CH^n with constant principal curvatures are realized as the tubes of constant radius over certain submanifolds when the structure vector field ξ is principal.

THEOREM B ([2]). *Let M be a connected real hypersurface of CH^n. Then M has constant principal curvatures and ξ is principal curvature vector if and only if M is locally congruent to one of the following;*

- (A_0) *a horosphere in CH^n,*
- (A_1) *a tube over a complex hyperbolic hyperplane CH^{n-1},*
- (A_2) *a tube over a totally geodesic CH^k $(1 \leq k \leq n-2)$,*
- (B) *a tube over a totally real hyperbolic space RH^n.*

On the other hand, it is known that there is no real hypersurface with parallel Ricci tensor $\nabla S = 0$ of $M^n(c)$, $c \neq 0$ ([4]). Because of this fact we know that there does not exist any Einstein real hypersurface of $M^n(c)$, $c \neq 0$. In such a situation, let us investigate the covariant derivative of the Ricci tensor in $M^n(c)$, $c \neq 0$, along the structure vector ξ in such a way that $\nabla_\xi S = 0$.

In order to prove our result we prepare the following theorems without proof:

THEOREM C ([5]). *Let M be a real hypersurface of CH^n. If the structure vector ξ is principal and $\nabla_\xi S = 0$, then M is locally congruent to one of* (A_0), (A_1) and (A_2).

THEOREM D ([8]). *Let M be a real hypersurface in $CP^n(\geq 3)$ on which ξ is a principal curvature vector and the focal map φ_r has constant rank on M. If $\nabla_\xi S = 0$, then M is locally congruent to one of* (A_1), (A_2), (B), (C), (D) and (E).

In this paper let us consider the condition that ξ is an eigenvector of the Ricci tensor S, which is more general notion than $A\xi = \alpha\xi$.

THEOREM. *Let M be a real hypersurface of $M^n(c)$, $c \neq 0$. If it satisfies $\nabla_\xi S = 0$ and $S\xi = \sigma\xi$ for some constant σ on M, then ξ is a principal curvature vector.*

All manifolds in this paper are assumed to be connected and of class C^∞ and the real hypersurfaces are supposed to be orientable.

1. Preliminaries

Let M be a real hypersurface of a complex n-dimensional complex space form $M^n(c)$ of constant holomorphic sectional curvature c, and let C be a unit normal vector field on a neighborhood of a point x in M. We denote by $\bar\nabla$ and ∇ the Riemannian connection in $M^n(c)$ and in M respectively. Then by the Gauss formula, we have the relationship between $\bar\nabla$ and ∇ : For any vector fields X and Y on M

$$\bar\nabla_X Y = \nabla_X Y + g(AX, Y)C,$$

where g is the Riemannian metric tensor of M induced from that of $M^n(c)$ and A denotes the shape operator with respect to C of M in $M^n(c)$. Furthermore, we have another equation which is called the Weingarten formula:

$$\bar\nabla_X C = -AX.$$

For any local vector field X on a neighborhood of x in M, the transformations of X and C under the complex structure J in $M^n(c)$ can be given by

$$JX = \phi X + \eta(X)C, \quad JC = -\xi,$$

where ϕ defines a skew-symmetric transformation on the tangent bundle TM of M, where η and ξ denote a 1-form and a vector field on a neighborhood of x in M respectively. Then it is seen that $g(\xi, X) = \eta(X)$. The set of tensors (ϕ, ξ, η, g) is called an *almost contact metric structure* on M. They satisfy the following

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0, \quad \eta(\xi) = 1,$$

where I denotes the identity transformation and $\otimes$ the tensor product. Furthermore the covariant derivatives of the structure tensors are given by

$$(1.1) \qquad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \quad \nabla_X \xi = \phi AX.$$

Since the ambient space is of constant holomorphic sectional curvature c, equations of the Gauss and Codazzi are respectively given as follows;

(1.2)
$$\begin{aligned} R(X,Y)Z = c\{&g(Y,Z)X - g(X,Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y \\ &- 2g(\phi X, Y)\phi Z\}/4 + g(AY, Z)AX - g(AX, Z)AY, \end{aligned}$$

$$(1.3) \quad (\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}/4,$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is a tensor of type (0,2) given by $S'(X,Y) = tr\{Z \to R(Z,X)Y\}$. Also it may be regarded as the tensor of type (1,1) and denoted by $S : TM \to TM$ satisfying $S'(X,Y) = g(SX, Y)$. From (1.3) we see that the Ricci tensor S of M is given by

$$(1.4) \qquad S = c\{(2n+1)I - 3\eta \otimes \xi\}/4 + hA - A^2,$$

where we have put $h = trA$. Moreover, using (1.2) we get

$$(1.5)\qquad \begin{aligned}(\nabla_X S)Y = &-3c\{g(\phi AX, Y)\xi + \eta(Y)\phi AX\}/4 \\ &+ dh(X)AY + (hI - A)(\nabla_X A)Y - (\nabla_X A)AY,\end{aligned}$$

where d denotes the exterior differential.

In what follows, to write our formulas in convention forms, we denote $\alpha = g(A\xi, \xi)$, $\beta = g(A^2\xi, \xi)$ and ∇f by the gradient vector field of a function f. Define a 1-form u by $u(X) = g(U, X)$, where $U = \nabla_\xi \xi$. Because of properties of the almost contact metric structure and the second equation of (1.1), we can get

$$(1.6)\qquad \phi U = -A\xi + \alpha\xi,$$

which shows that $g(U, U) = \beta - \alpha^2$. By the definition of U and the second equation of (1.1), we easily see that

$$(1.7)\qquad g(\nabla_X \xi, U) = g(A^2\xi, X) - \alpha g(A\xi, X).$$

On the other hand, differentiating (1.6) covariantly and making use of (1.1), we find

$$(1.8)\qquad \begin{aligned}\eta(X)g(AU, Y) + g(\phi X, \nabla_Y U) = &g((\nabla_Y A)X, \xi) - g(A\phi AX, Y) \\ &- \eta(X)g(\nabla\alpha, Y) + \alpha g(A\phi X, Y),\end{aligned}$$

which enables us to obtain

$$(1.9)\qquad g((\nabla_X A)\xi, \xi) = 2g(AX, U) + g(\nabla\alpha, X).$$

By the definition of U, (1.1), (1.8) and (1.9) it is verified that

$$(1.10)\qquad \nabla_\xi U = 3\phi AU + \alpha A\xi - \beta\xi + \phi\nabla\alpha.$$

2. Real hypersurfaces of $M^n(c)$ satisfying $\nabla_\xi S = 0$

In what follows let M be a real hypersurface of $M^n(c)$, $c \neq 0$ and assume that the Ricci tensor S satisfies

$$\nabla_\xi S = 0, \tag{2.1}$$

and that

$$S\xi = \sigma\xi \tag{2.2}$$

for some function σ. Then by (1.4) we have

$$A^2\xi = hA\xi + (\beta - h\alpha)\xi, \tag{2.3}$$

where we have put

$$\beta - h\alpha = -\sigma + \frac{c}{2}(n-1). \tag{2.4}$$

Differentiating (2.2) covariantly along M, we find

$$(\nabla_X S)\xi + S\nabla_X\xi = (X\sigma)\xi + \sigma\nabla_X\xi.$$

Because of (2.1) we then obtain

$$SU = d\sigma(\xi)\xi + \sigma U,$$

which together with (2.2) gives

$$d\sigma(\xi) = 0, \tag{2.5}$$

and hence $SU = \sigma U$. It follows that

$$A^2U = hAU + \left(\beta - h\alpha + \frac{3}{4}c\right)U, \tag{2.6}$$

where we have used (1.4) and (2.4).

We put $A\xi = \alpha\xi + \mu W$, where W is a unit vector field orthogonal to ξ. Then from (1.6) we see that $U = \mu\phi W$, and W is also orthogonal to U. We assume that $\mu \neq 0$ on M, that is, ξ is not a principal curvature vector and we put $\Omega = \{p \in M | \mu(p) \neq 0\}$. Then Ω is an open subset of M and from now on we discuss our arguments on Ω.

Making use of (2.3), we find

$$AW = (h - \alpha)W + \mu\xi, \tag{2.7}$$

where $\mu^2 = \beta - \alpha^2$ and hence

$$A^2W - hAW = (\beta - h\alpha)W$$

because of $\mu \neq 0$.

If we differentiate (2.3) covariantly along Ω and use the second equation of (1.1), then we get

$$\begin{aligned} &(\nabla_X A)A\xi + A(\nabla_X A)\xi + A^2\phi AX - hA\phi AX \\ &= dh(X)A\xi + h(\nabla_X A)\xi + d(\beta - h\alpha)(X)\xi + (\beta - h\alpha)\phi AX. \end{aligned} \tag{2.8}$$

So, by using (1.9) and (2.3), we obtain

$$g((\nabla_X A)\xi, A\xi) = hg(AU, X) + \frac{1}{2}d\beta(X), \tag{2.9}$$

which together with (1.3) gives

$$g((\nabla_\xi A)AX, \xi) = hg(AU, X) - \frac{c}{4}u(X) + \frac{1}{2}d\beta(X). \tag{2.10}$$

Replacing X by ξ in(2.8) and using (1.9) and (2.10), we get

$$hAU + 2(\beta - h\alpha + c)U = dh(\xi)A\xi - A\nabla\alpha + h\nabla\alpha - \frac{1}{2}\nabla\beta. \tag{2.11}$$

If we take the inner product with (2.11) and ξ and make use of (2.4) and (2.5), then we can derive the equation

$$2g(A\xi, \nabla\alpha) = \alpha dh(\xi) + hd\alpha(\xi),$$

which implies

$$2\mu d\alpha(W) = (h - 2\alpha)d\alpha(\xi) + \alpha dh(\xi). \tag{2.12}$$

Since $\nabla_\xi S = 0$, by replacing X by ξ, we have from (1.3) and (1.5),

$$\begin{aligned} &\frac{3}{4}c\{u(X)\eta(Y) + u(Y)\eta(X)\} + \frac{c}{4}\{g(AY, \phi X) + g(AX, \phi Y)\} \\ &= dh(\xi)g(AX, Y) + hg((\nabla_X A)Y, \xi) + \frac{c}{4}hg(\phi X, Y) \\ &- g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi). \end{aligned} \tag{2.13}$$

By replacing X with $A\xi$ in (2.13) and using (1.9) and (2.6), we find

$$\begin{aligned} &\left(h^2 + 2\beta - 2h\alpha - \frac{c}{4}\right) AU + \left\{h\beta - h^2\alpha + \frac{3}{4}c(h + \alpha)\right\} U \\ &= dh(\xi)A^2\xi - \frac{1}{2}A\nabla\beta - (\beta - h\alpha)\nabla\alpha. \end{aligned} \tag{2.14}$$

Using (2.11) and (2.14) we have the following:

$$\frac{3}{4}c(3AU - \alpha U) + A^2\nabla\alpha - hA\nabla\alpha - (\beta - h\alpha)\nabla\alpha = 0, \tag{2.15}$$

$$\begin{aligned} &\frac{1}{2}(A\nabla\beta - h\nabla\beta) - h(A\nabla\alpha - h\nabla\alpha) + (\beta - h\alpha)\nabla\alpha \\ &= dh(\xi)(\beta - h\alpha)\xi + \left(2h\alpha - 2\beta + \frac{c}{4}\right) AU \\ &+ \left(h\beta - h^2\alpha + \frac{5}{4}ch - \frac{3}{4}c\alpha\right) U. \end{aligned} \tag{2.16}$$

Combining (2.11) and (2.15) with (2.16), we can verify that

$$\begin{aligned} &-\frac{1}{2}\{A^2\nabla\beta - hA\nabla\beta - (\beta - h\alpha)\nabla\beta\} \\ &= \frac{3}{4}c\{(h + \alpha)AU - (\beta + \frac{c}{4})U\}. \end{aligned} \tag{2.17}$$

Now, differentiating (2.7) covariantly along Ω, we find

$$(\nabla_X A)W + A\nabla_X W = d\mu(X)\xi + \mu\nabla_X\xi + d(h-\alpha)(X)W + (h-\alpha)\nabla_X W.$$

By taking the inner product in the last equation with W , we have

$$g((\nabla_X A)W, W) = -2g(AX, U) + dh(X) - d\alpha(X). \tag{2.18}$$

because W is a unit vector field orthogonal to ξ. We also have by applying ξ

$$\mu g((\nabla_X A)W, \xi) = (h-2\alpha)g(AU, X) + \frac{1}{2}d\beta(X) - \alpha d\alpha(X). \tag{2.19}$$

If we replace X by μW to the both sides of (2.13) and take account of (1.3), (2.6) and (2.19), then we obtain

$$\begin{aligned}&\left(h^2 - 3\alpha h + 2\beta - \frac{c}{4}\right) AU + \left\{(h-2\alpha)\left(\beta - h\alpha + \frac{3}{4}c\right) + \frac{c}{4}\alpha\right\}U\\ &= \mu dh(\xi)AW - \mu A\nabla\mu - \beta\nabla\alpha + \frac{1}{2}\alpha\nabla\beta.\end{aligned} \tag{2.20}$$

On the other hand, we have from (1.3) and (2.8)

$$\begin{aligned}&\frac{c}{4}\{-u(X)\eta(Y) + u(Y)\eta(X)\} + \frac{c}{2}(h-\alpha)g(\phi Y, X)\\ &\quad - g(A^2\phi AX, Y) + g(A^2\phi AY, X) + 2hg(\phi AX, AY)\\ &\quad - (\beta - h\alpha)\{g(\phi AY, X) - g(\phi AX, Y)\}\\ &= g(AY, (\nabla_X A)\xi) - g(AX, (\nabla_Y A)\xi) + dh(Y)g(A\xi, X)\\ &\quad - dh(X)g(A\xi, Y) + d(\beta - h\alpha)(Y)\eta(X) - d(\beta - h\alpha)(X)\eta(Y).\end{aligned} \tag{2.21}$$

Thus we get

$$\begin{aligned}&\left(4\beta - 4h\alpha + h^2 + \frac{c}{4}\right) AU + \left(\frac{3}{2}c\alpha - \frac{5}{4}ch\right) U\\ &= \mu dh(\xi)AW - \mu dh(W)A\xi - \mu\{d\beta(W) - hd\alpha(W) - \alpha dh(W)\}\xi\\ &\quad + \frac{1}{2}(2\alpha - h)\nabla\beta + (h\alpha - 2\beta)\nabla\alpha + (\beta - \alpha^2)\nabla h,\end{aligned} \tag{2.22}$$

where we have used (1.9), (2.6), (2.7), (2.19) and (2.20), which implies

$$\begin{aligned}\frac{3}{4}\left(4\beta-4h\alpha+h^2+\frac{c}{4}\right)AU+\left(\frac{3}{2}c\alpha-\frac{5}{4}ch\right)U\\=\frac{1}{2}(2\alpha-h)\{A^2-hA-(\beta-h\alpha)I\}\nabla\beta\\+(\alpha h-2\beta)\{A^2-hA-(\beta-h\alpha)I\}\nabla\alpha\\+(\beta-\alpha^2)\{A^2-hA-(\beta-h\alpha)I\}\nabla h.\end{aligned}$$

Thus, if we take account of (2.15) and (2.17), then we obtain

(2.23)

$$\begin{aligned}(\beta-\alpha^2)\{A^2-hA-(\beta-h\alpha)I\}\nabla h\\=\frac{3}{4}c\left\{\left(-2\beta+2\alpha^2+\frac{c}{4}\right)AU+(h\beta-h\alpha^2+c\alpha-ch)U\right\},\end{aligned}$$

which enables us to obtain

$$\begin{aligned}g(U,U)dh(U)=\left(-2\beta+2\alpha^2+\frac{c}{4}\right)g(AU,U)\\+(h\beta-h\alpha^2+c\alpha-ch)g(U,U),\end{aligned}\tag{2.24}$$

because of (2.6).

3. The case that σ is constant

In this section, we assume that M is a real hypersurface of $M^n(c)$, $c\neq 0$ satisfying (2.1) and (2.2) with $\sigma=$ constant. Then by (2.4) we have

(3.1,) $$\nabla\beta=h\nabla\alpha+\alpha\nabla h.$$

Thus, using (2.15), (2.17) and (2.23) we obtain

(3.2)

$$\left(h\beta-h\alpha^2-\frac{1}{4}c\alpha\right)AU=\left\{2(\beta-\alpha^2)\left(h\alpha-\beta-\frac{3}{4}c\right)+c(\beta-h\alpha)\right\}U.$$

Applying (2.14) by ξ and making use of (3.1), we find

$$\mu d\beta(W) = (2\beta - \alpha^2)dh(\xi) + (\alpha h - 2\beta)d\alpha(\xi),$$

or using (2.12) and (3.1),
(3.3)

$$\mu\alpha dh(W) = \left(2\beta - \alpha^2 - \frac{1}{2}\alpha h\right) dh(\xi) + \left(2\alpha h - 2\beta - \frac{1}{2}h^2\right) d\alpha(\xi).$$

Now, let Ω_1 be the set of points in Ω such that U is not principal. Then we have on Ω_1

$$h(\beta - \alpha^2) = \frac{c}{4}\alpha, \quad 2(\beta - \alpha^2)\left(h\alpha - \beta - \frac{3}{4}c\right) + c(\beta - h\alpha) = 0. \tag{3.4}$$

Differentiation the second equation of (3.4) along Ω_1 gives $\nabla\beta - 2\alpha\nabla\alpha = 0$, which shows that $(h - 2\alpha)\nabla\alpha + \alpha\nabla h = 0$.

In a similar way, from the first equation of (3.4), we have $(\beta - \alpha^2)\nabla h = \frac{c}{4}\nabla\alpha$ on Ω_1. Thus by virtue of these two equations we have $(\beta - \alpha^2)(2\alpha - h)\nabla\alpha = \frac{c}{4}\alpha\nabla\alpha$. Therefore we have $\nabla\alpha = 0$ on Ω_1 because of (3.4). So that (2.15) implies $3AU = \alpha U$ on Ω_1. This can't occur in Ω_1.

Hence (3.2) means $AU = \lambda U$ on Ω, and so we have

$$\left\{h(\beta - \alpha^2) - \frac{c}{4}\alpha\right\}\lambda = 2(\beta - \alpha^2)\left(h\alpha - \beta - \frac{3}{4}c\right) + c(\beta - h\alpha).$$

By using $AU = \lambda U$ and (2.6), we find

$$\lambda^2 = \lambda h + \beta - h\alpha + \frac{3}{4}c. \tag{3.5}$$

Therefore, the last two equations imply

$$\lambda(h - 2\lambda)(\beta - \alpha^2) = \frac{c}{4}(4h\alpha - 4\beta - \alpha\lambda). \tag{3.6}$$

Thus we have

LEMMA 1. *$AU = \lambda U$ on Ω, where λ satisfies* (3.5) *and* (3.6).

From this lemma and (2.24), it follows that

$$dh(U) = \left(\beta - \alpha^2 - \frac{c}{4}\right)(h - 2\lambda) + \frac{c}{4}(4\alpha - \lambda - 3h). \tag{3.7}$$

Therefore we obtain, by using that σ is constant and with (2.15), (2.17) and Lemma 1,

$$\alpha dh(U) = \left(h\lambda - 2\alpha\lambda + 2\beta - h\alpha + \frac{c}{2}\right)(\beta - \alpha^2). \tag{3.8}$$

The covariant differentiation of (3.5) gives

$$(2\lambda - h)\nabla\lambda = \lambda\nabla h. \tag{3.9}$$

We notice here that $\lambda \neq 0$ on Ω because of (3.5) and (3.6). So $\alpha \neq 0$ on Ω since we have (2.15). Consequently, we can, using (3.6), (3.7) and (3.9), verify that $2\lambda - h \neq 0$ on Ω.

On the other hand, if we make use of (2.13) and Lemma 1, then we get

(3.10)
$$\begin{aligned}&(h - \lambda)g((\nabla_X A)U, \xi) - g(AX, (\nabla_U A)\xi)\\ &= \frac{3}{4}cg(U,U)\eta(X) + \frac{c}{4}\mu(\lambda - h)w(X) - \frac{c}{4}\mu g(AX, W) - \lambda dh(\xi)u(X).\end{aligned}$$

In a same way, from (2.21) and (3.5) we have

$$\begin{aligned}&\lambda g((\nabla_X A)U, \xi) - g(AX, (\nabla_U A)\xi) + dh(U)g(AX, \xi)\\ &= \frac{c}{4}g(U,U)\eta(X) - \frac{c}{2}(h - \alpha)\mu w(X) - \frac{3}{4}c\mu g(AX, W).\end{aligned}$$

From the above two equations we obtain

$$\begin{aligned}&(h - 2\lambda)g((\nabla_X A)U, \xi) - dh(U)g(AX, \xi)\\ &= \frac{c}{2}g(U,U)\eta(X) + \frac{c}{2}\mu g(AX, W) + \frac{c}{2}(h - \alpha)\mu w(X)\\ &\quad + \frac{c}{4}\mu(\lambda - h)w(X) - \lambda dh(\xi)u(X).\end{aligned} \tag{3.11}$$

If we differentiate $AU = \lambda U$ covariantly along Ω, then we get

(3.12) $$(\nabla_X A)U + A(\nabla_X U) = d\lambda(X)U + \lambda\nabla_X U,$$

which by taking the inner product with ξ and using (1.3) and (1.10) implies

(3.13)
$$\begin{aligned} g((\nabla_X A)U, \xi) = d\lambda(\xi)u(X) - \frac{c}{4}\mu w(X) - g(AX, \phi\nabla\alpha) - \lambda g(\phi X, \nabla\alpha) \\ + \mu(3\lambda - \alpha)\{g(AX, W) - \lambda w(X)\} \\ + (\beta - \alpha^2)\{g(AX, \xi) - \lambda\eta(X)\}. \end{aligned}$$

Combining (3.11) with (3.13) and taking account of (3.5) and (3.7), we have

(3.14) $$A\phi\nabla\alpha + \lambda\phi\nabla\alpha = -\mu(3\lambda - \alpha)(AW - \lambda W).$$

Thus (3.13) turns out to be

(3.15)
$$g((\nabla_X A)U, \xi) = d\lambda(\xi)u(X) + (\beta - \alpha^2)\{g(AX, \xi) - \lambda\eta(X)\} - \frac{c}{4}\mu w(X).$$

Since $\nabla_X\xi = \phi AX$ and $U = \nabla_\xi\xi$, we see that $\nabla_X U = \phi(\nabla_X A)\xi + \alpha AX - g(A^2X, \xi)\xi$. Replacing X by U and using (1.3) and (3.15), we have

(3.16) $$\nabla_U U = -\lambda(h - \alpha)U + \left(\alpha\lambda + \beta - \alpha^2 + \frac{c}{4}\right)U - \mu d\lambda(\xi)W.$$

On the other hand we find from (3.12)

$$\begin{aligned} &\frac{c}{4}\{\eta(Y)\phi X - \eta(X)\phi Y\}U + g(AX, \nabla_Y U) - g(AY, \nabla_X U) \\ &= d\lambda(Y)u(X) - d\lambda(X)u(Y) + \lambda\{(\nabla_Y u)(X) - (\nabla_X u)(Y)\}, \end{aligned}$$

which shows that by using (3.16) and Lemma 1,

$$\mu d\lambda(\xi)(AW - \lambda W) = g(U, U)\nabla\lambda - d\lambda(U)U.$$

It follows that

(3.17) $$\mu dh(\xi)(AW - \lambda W) = (\beta - \alpha^2)\nabla h - dh(U)U$$

because of (3.9). Therefore we have

(3.18) $$\mu dh(W) = (h - \alpha - \lambda)dh(\xi).$$

We now prove

LEMMA 2. *$dh(\xi) = 0$ and $d\alpha(\xi) = 0$ on Ω.*

Proof. From $\mu^2 = \beta - \alpha^2$ and (3.1) we have

$$2\mu\nabla\mu = \alpha\nabla h + (h - 2\alpha)\nabla\alpha. \qquad \square$$

Differentiating (3.6) covariantly along Ω and making use of (3.9), we find

$$2\lambda(h - 2\lambda)\mu\nabla\mu - 2\mu^2\lambda\nabla\lambda = -\frac{c}{4}(\lambda\nabla\alpha + \alpha\nabla\lambda).$$

As is already remarked that $\lambda(h-2\lambda) \neq 0$ on Ω, the last two equations imply that

$$x\nabla h + y\nabla\alpha = 0, \tag{3.19}$$

where we have put

$$\begin{cases} x = \alpha(h - 2\lambda)^2 + 2\lambda\mu - \frac{c}{4}\alpha, \\ y = (h - 2\lambda)\{(h - 2\lambda)(h - 2\alpha) + \frac{c}{4}\}. \end{cases}$$

So we have $\{y\alpha - x(2\lambda - h)\}dh(\xi) = 0$ because of (2.12) and (3.18). Let Ω_2 be the set of points at which $dh(\xi) \neq 0$ in Ω. Suppose that Ω_2 is not empty. Then we have $y\alpha = x(2\lambda - h)$ on Ω_2. Therefore we have $x(\lambda h + 2\beta - 2h\alpha + 2c) = 0$ on Ω_2 because of (2.15), (2.17), (3.1), (3.5), (3.9) and (3.19). Let $\Omega_3 = \{p \in \Omega_2 | x(p) \neq 0\}$. Suppose that Ω_3 is nonvoid. Then we have $\lambda h + 2\beta - 2h\alpha + 2c = 0$ on Ω_3. So we have λh = constant because of (2.4). From this fact and (3.9) we see that $\nabla h = 0$ on Ω_3, a contradiction. Therefore x=0 on Ω_2 and hence y=0 because $\alpha \neq 0$ on Ω. Thus (3.19) leads to

$$(h - 2\lambda)(h - 2\alpha) + \frac{c}{4} = 0, \tag{3.20}$$

which enables us to obtain

$$(2\lambda - h)^2\nabla\alpha = \left(h^2 + 2\beta - 3\alpha h + \frac{3}{2}c\right)\nabla h, \tag{3.21}$$

on Ω_2. If we take the inner product with W and make use of (3.18), then we get

$$\mu d\alpha(W) = (h - \alpha - \lambda) d\alpha(\xi)$$

because $2\lambda - h \neq 0$ on Ω, or use (2.12)

$$\alpha dh(\xi) = (h - 2\lambda) d\alpha(\xi) \tag{3.22}$$

on Ω_2. Therefore we have $d\alpha(\xi) \neq 0$ on Ω_2.

Applying ξ to (3.21) and using (3.20) and (3.22), we find $\lambda^2 - \alpha\lambda - \frac{c}{8} = 0$ on Ω_2. From this and (3.9) and (3.22) we see that $\lambda = 0$ on Ω_2, a contradiction. This is impossible on Ω. Thus $dh(\xi) = 0$ on Ω. So (3.18) means $dh(W) = 0$ and hence $\{(h - 2\alpha)^2 + 4(\beta - \alpha^2)\} d\alpha(\xi) = 0$ because of (3.3). Therefore $d\alpha(\xi) = 0$ on Ω. This completes the proof of Lemma 2.

According to Lemma 2, (3.17) turns out to be

$$g(U, U)\nabla h = dh(U)U,$$

or using (3.8),

$$\alpha \nabla h = \left(h\lambda - 2\alpha\lambda + 2\beta - h\alpha + \frac{c}{2}\right) U. \tag{3.23}$$

By differentiating (3.23) covariantly, we have

$$\begin{aligned}
&d\alpha(Y)dh(X) - d\alpha(X)dh(Y) \\
&- \{(h - 2\alpha)d\lambda(Y) + (\lambda + \alpha)dh(Y) + (h - 2\lambda)d\alpha(Y)\}u(X) \\
&+ \{(h - 2\alpha)d\lambda(X) + (\lambda + \alpha)dh(X) + (h - 2\lambda)d\alpha(X)\}u(Y) \\
&- \left(h\lambda - 2\alpha\lambda + 2\beta - h\alpha + \frac{c}{2}\right) \{(\nabla_Y u)X - (\nabla_X u)Y\} \\
&= 0.
\end{aligned}$$

Hence we have

$$\left(h\lambda - 2\alpha\lambda + 2\beta - h\alpha + \frac{c}{2}\right) du(\xi, X) = 0$$

for any vector field X because of (3.8) and Lemma 2. Let $M_0 = \{p \in \Omega | du(\xi, X)(p) \neq 0\}$. Suppose that M_0 is not empty. Then on a component C of M_0 we have

$$h\lambda - 2\alpha\lambda + 2\beta - h\alpha + \frac{c}{2} = 0. \tag{3.24}$$

Thus (3.23) implies $\nabla h = 0$ so that (3.7) becomes

$$(h - 2\lambda)\left(\beta - \alpha^2 - \frac{c}{4}\right) - \frac{c}{4}(\lambda - h) - c(h - \alpha) = 0 \tag{3.25}$$

on C. By means of (3.9), we see that λ is constant on C. By using (3.1) and (3.24) it is seen that α is constant on C. Thus (2.15) and (2.17) turn out to be $3\lambda = \alpha, (h + \alpha)\lambda = \beta + \frac{c}{4}$ on C respectively . So these facts, (3.5) and (3.25) will produce a contradiction. Hence M_0 is void.

Therefore we have

LEMMA 3. *$du(\xi, X) = 0$ for any vector field X on Ω.*

4. Proof of Theorem

Using Lemma 3 and (2.7) and the definition of U we have

$$\nabla_\xi U = -\mu\{\mu\xi + (h - \alpha)W\}.$$

Hence, from (1.10) and Lemma 1 we see that

$$\mu(h - \alpha)W = -\mu(\alpha - 3\lambda)W - \phi\nabla\alpha, \tag{3.26}$$

which implies $\mu^2(h - \alpha) = -\mu^2(\alpha - 3\lambda) + d\alpha(U)$. From (2.6), (2.15) and Lemma 1 we can get $h = \alpha$. Thus it is clear that

$$\phi\nabla\alpha = \mu(3\lambda - \alpha)W.$$

Accordingly we obtain

$$\nabla\alpha = -(3\lambda - \alpha)U. \tag{3.27}$$

Substituting $h = \alpha$ into (3.5), we have $\lambda^2 = \beta - \alpha^2 + \lambda\alpha + \frac{3}{4}c$ and comparing (3.23) with (3.27), we get

$$\beta - \alpha^2 + \frac{c}{4} + \alpha\lambda = 0.$$

From these two equations, (3.9) and (3.27) we deduce a contradiction. Hence we conclude that Ω is empty. It completes the proof of main theorem.

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