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WEAKLY LAGRANGIAN EMBEDDING AND PRODUCT MANIFOLDS

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ABSTRACT. We investigate when the product of two smooth manifolds admits a weakly Lagrangian embedding. We prove that, if M^m and N^n are smooth manifolds such that M admits a weakly Lagrangian embedding into $\mathbb{C}^m$ whose normal bundle has a nowhere vanishing section and N admits a weakly Lagrangian immersion into $\mathbb{C}^n$, then $M \times N$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$. As a corollary, we obtain that $S^m \times S^n$ admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ if $n = 1,\ 3$. We investigate the problem of whether $S^m \times S^n$ in general admits a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$.

1. Introduction

The notion of weakly Lagrangian embedding was introduced by T. Kawashima ([5]) as a weaker version of Lagrangian embedding. He showed that S^n admits a weakly Lagrangian embedding into $\mathbb{C}^n$ if and only if $n = 1, 3$, from which it follows that S^n does not admit any Lagrangian embedding into $\mathbb{C}^n$ if $n \neq 1, 3$. In fact, later it has been shown that, for any manifold M^n which admits a Lagrangian embedding into $\mathbb{C}^n$, we have $\pi_1(M) \neq 1$ ([2]). Therefore it follows that S^n admits a Lagrangian embedding into $\mathbb{C}^n$ only when $n = 1$.

This note investigates when the product of two smooth manifolds admits a weakly Lagrangian embedding. In particular, we have

THEOREM 1. *Let M, N be smooth manifolds of dimension m, n, respectively. Assume that M admits a weakly Lagrangian embedding into $\mathbb{C}^m$ whose normal bundle has a nowhere vanishing section and N*

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admits a weakly Lagrangian immersion into $\mathbb{C}^n$. *Then* $M \times N$ *admits a weakly Lagrangian embedding into* $\mathbb{C}^{m+n}$.

In fact, the assumption on the existence of a nowhere vanishing section on the normal bundle is redundant if M is an oriented closed manifold: Let $f : M \to \mathbb{C}^m$ be a weakly Lagrangian embedding. We have that $\nu_f \cong (-1)^{n(n-1)/2}TM$ (Proposition 2.1) and $\chi(M) \doteq 0$ (Lemma 4.1). Thus the Euler characteristic of ν_f vanishes, which means ν_f admits a nowhere vanishing section.

As a corollary of Theorem 1, we conclude:

THEOREM 2. $S^m \times S^n$ *admits a weakly Lagrangian embedding into* $\mathbb{C}^{m+n}$ *if* n *is* 1 *or* 3.

In particular, the above provides more examples, in addition to S^3, of manifolds which admits a weakly Lagrangian embedding but not any Lagrangian embedding (see Corollary 3.2 below).

Also we have that $S^m \times S^n$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ if both m and n are even (see the below of Lemma 4.1). However we don't know what happens when one of m, n is odd while none of the two is 1 or 3, which is a subject of our ongoing investigation. We will provide a reason why this problem is more difficult in this case in the last section.

2. Basic notions and facts

Two subbundles η_0 and η_1 of a vector bundle ξ over a smooth manifold M is said to be *homotopic* if there exists a subbundle $\tilde{\eta}$ of $\xi \times I$ such that $\tilde{\eta}|_{M\times\{0\}} = \eta_0$ and $\tilde{\eta}|_{M\times\{1\}} = \eta_1$.

A *symplectic form* on a vector bundle is a nondegenerate two form on it. A vector bundle of finite rank is referred to as a *Lagrangian vector bundle* if it is considered with a fixed symplectic two form. Note that a Lagrangian vector bundle should be of even rank. A subbundle η of a Lagrangian vector bundle ξ is a *Lagrangian subbundle* if 2 (rank η) $=$ rank ξ and the restriction of the symplectic form to η is the zero form. A subbundle η of a symplectic vector bundle ξ is called a *weakly Lagrangian subbundle* if η is homotopic to a Lagrangian subbundle of ξ.

Now let $f : L \to M$ be an embedding (resp. immersion) of a smooth manifold L into a symplectic manifold M with a symplectic structure ω. We call f a *Lagrangian embedding* (resp. *immersion*) if the tangent bundle TL of L is a Lagrangian subbundle of the symplectic vector bundle f^*TM (with the symplectic form $f^*\omega$). Similarly, f is a *weakly Lagrangian embedding* (resp. *immersion*) if TL is a weakly Lagrangian subbundle of f^*TM.

We will consider $\mathbb{C}^n$ with the usual symplectic structure. A Lagrangian embedding or a weakly Lagrangian embedding will be understood as 'into $\mathbb{C}^n$' unless otherwise specified.

Note that the notion of weakly Lagrangian embedding (resp. immersion) is invariant under regular homotopy. That is, if f_0 and f_1 are embeddings (resp. immersions) homotopic through embeddings (resp. immersions) and f_0 is a weakly Lagrangian embedding (resp. immersion), then f_1 is also such.

We recall some basic properties of a weakly Lagrangian embedding.

PROPOSITION 2.1. *For a weakly Lagrangian embedding $f : L^n \to M^{2n}$, from an oriented manifold L, the followings hold*

i) $\nu(f) \cong (-1)^{n(n-1)/2}TL$, *as oriented vector bundles, where $\nu(f)$ is the normal bundle of f with orientation defined in the usual way.*

ii) *If L is a closed manifold and $a = f_*([L]) \in H_n(M, Z)$, then we have*

$$a \cdot a = (-1)^{n(n-1)}\chi(L)$$

where $[L] \in H_n(L, Z)$ denotes the fundamental class and $a \cdot a$ is the Kronecker index $\langle Da, a\rangle$ with D denoting the Poincaré isomorphism $H_n(M, Z) \to H^n_{comp}(M, Z)$.

The proof is a copy of that of Proposition 2, [5]. We note that i) above is true even if f is only a weakly Lagrangian *immersion*. On the other hand, we need the condition that f is an embedding for ii) above since in this case we make use of the normal neighborhood of $f(L) \subset M$, which is impossible if f is just an immersion.

3. Proofs of Theorem 1, 2

The following is the key lemma to prove THEOREM 1.

LEMMA 3.1. *Assume* $f : M^m \to P^{2m}$, $g : N^n \to Q^{2n}$ *are maps between smooth manifolds such that* i) f *is an embedding whose normal bundle has a nowhere vanishing section and* ii) g *is an immersion. Then* $f \times g : M \times N \to P \times Q$ *is regularly homotopic to an embedding.*

Proof. We may assume that g is completely regular (cf. [1]). Let $y_1, y_2, \cdots$ and $z_1, z_2, \cdots$ be distinct points in N such that $g(y_i) = g(z_i)$, $i = 1, 2, \cdots$. Note that such points appear discretely.

We may construct (for example, using the exponential map) neighborhoods $U_1, U_2, \cdots$ of $y_1, y_2, \cdots$ which are diffeomorphic to the closed disc D^n and such that $U_i \cap U_j = \phi$ if $i \neq j$ and $U_i \cap \{y_1, y_2, \cdots, z_1, z_2, \cdots\} = \{y_i\}$, $i = 1, 2, \cdots$.

Let $\delta : N \to [0,1]$ be a smooth function such that $\delta(y_i) = 1$, $i = 1, 2, \cdots$ and $\delta(N - \cup_{i=1,2,\cdots} U_i) = \{0\}$.

Note that the existence of nowhere vanishing section of the normal bundle is equivalent to the existence of a smooth embedding $F : M \times [0,1] \to P$ such that $F(x, 0) = f(x)$.

Now consider the map

$$H : M \times N \times [0,1] \to P \times Q$$

defined by $H(x, y, t) = (F(x, t\delta(y)), g(y))$.

It is straightforward to see that for each $t \in [0,1]$, $H_t : M \times N \to P \times Q$ is an immersion. Thus H_0, H_1 are regularly homotopic to each other.

We show that H_1 is an embedding as follows : Assume that $H_1(x, y) = H_1(x', y')$, that is, $F(x, \delta(y)) = F(x', \delta(y'))$ and $g(y) = g(y')$, while $(x, y) \neq (x', y')$. If $y = y'$, then we have $F(x, \delta(y)) = F(x', \delta(y))$ and we may conclude $x = x'$ since F is an embedding. Therefore, we have $y \neq y'$. Now, by assumption on g, $g(y) = g(y')$ implies that $y = y_i, y' = z_i$ (or $y = z_i, y' = y_i$) for some i. But then we have $\delta(y_i) = 1$, $\delta(z_i) = 0$ and $F(x, \delta(y)) = F(x', \delta(y'))$ is impossible since F is an embedding. This proves the Lemma. □

As corollaries of the previous lemma, we obtain

Proof of Theorem 1. Let $f : M \to \mathbb{C}^m, g : N \to \mathbb{C}^n$ be the weakly Lagrangian embedding and the weakly Lagrangian immersion, respectively. Then $f \times g : M \times N \to \mathbb{C}^m \times \mathbb{C}^n = \mathbb{C}^{m+n}$ is regularly homotopic to an embedding by the previous lemma. Since being a weakly Lagrangian immersion is invariant under regular homotopy, the proof is complete. □

Proof of Theorem 2. According to Kawashima ([5]), S^n admits a weakly Lagrangian embedding if and only if $n = 1, 3$. Also according to Weinstein ([6]), S^n admits a Lagrangian immersion for any natural number n. □

COROLLARY 3.2. *$S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if $n_i = 1$ or 3 for some $i = 1, 2, \cdots, k$.*

Note that $S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}$ admits a weakly Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$, but it does not admit any Lagrangian embedding into $\mathbb{C}^{n_1+n_2+\cdots+n_k}$ if $n_i = 3$ for some i and $n_i \neq 1$ for any $i = 1, 2, \cdots k$, since in this case $\pi_1(S^{n_1} \times S^{n_2} \times \cdots \times S^{n_k}) = 1$.

4. The case of $S^m \times S^n$

As a corollary of ii), Proposition 2.1 we have the following.

LEMMA 4.1. *Let L be an orientable compact smooth n-manifold which admits a weakly Lagrangian embedding into $\mathbb{C}^n$. Then we have $\chi(L) = 0$.*

Lemma 4.1 proves that, if both m, n are even, $S^m \times S^n$ does not admit any weakly Lagrangian embedding since $\chi(S^m \times S^n) \neq 0$. In fact, the same result can also be obtained by i) of Proposition 2.1 since the tangent bundle of $S^m \times S^n$ is non-trivial if both m, n are even, while the normal bundle of *any embedding* of $S^m \times S^n$ is trivial if $m, n > 1$, which follows from the triviality of the normal bundle of the standard embedding of $S^m \times S^n$ into $\mathbb{C}^{m+n}$ and also from the following.

LEMMA 4.2. *For any simply connected closed smooth m-manifold, $m \geq 4$, any two of its embeddings into $\mathbb{C}^m$ are isotopic to each other through smooth embeddings.*

Note that if two embeddings are isotopic to each other then the normal bundles of them are isomorphic. A proof of Lemma 4.2 is provided below in this section. Note that S^2 is the only simply connected 2-manifold and it does not admit any weakly Lagrangian embedding and also that any compact orientable 3-manifold is parallelizable. Therefore, we may summarize and generalize the discussions so far as follows.

PROPOSITION 4.3. *Let M be a simply connected closed smooth m-manifold which admits an embedding into $\mathbb{C}^m$ whose normal bundle is trivial. If M admits a weakly Lagrangian embedding into $\mathbb{C}^m$, then TM is trivial.*

Note that, if the tangent bundle of a manifold is trivial, its Euler characteristic vanishes even if the converse is not true in general. Therefore we have obtained a sharper condition than the vanishing of the Euler characteristic for $S^m \times S^n$ to admit a weakly Lagrangian embedding into $\mathbb{C}^{m+n}$; its tangent bundle should be trivial.

However, we are not lucky enough here as the following holds.

FACT. The tangent bundle of $S^m \times S^n$ is trivial if m or n is odd.

Therefore, even if neither of m, n is 1 nor 3, we cannot conclude that $S^m \times S^n$ does not admit any weakly Lagrangian embedding into $\mathbb{C}^{m+n}$ if one of m, n is odd. The problem is left open.

The above fact follows from the observation below.

Let M be a smooth m-manifold such that

i) the tangent bundle TM is stably trivial and

ii) $TM \cong \xi + \epsilon^1_M$ for some vector bundle ξ over M of rank $m-1$.

Here ϵ^1_M means the trivial vector bundle of rank 1 and $\xi + \epsilon^1_M$ means the Whitney sum.

Let N denote another smooth n-manifold whose tangent bundle is stably trivial and consider the product manifold $M \times N$.

OBSERVATION. $T(M \times N) \cong \epsilon^{m+n}_{M\times N}$.

Proof. It is well-known that

$$T(M \times N) \cong TM \times TN.$$

By the assumption,

$$TM \times TN \cong (\xi + \epsilon^1_M) \times TN.$$

Let p_1, p_2 denote the projections from $M \times N$ to M, N, respectively. Then we have

$$(\xi + \epsilon_M^1) \times TN = p_1^*(\xi + \epsilon_M^1) + p_2^*TN \cong p_1^*\xi + p_1^*\epsilon_M^1 + p_2^*TN.$$

Now it is straightforward to see that

$$p_1^*\xi + p_1^*\epsilon_M^1 + p_2^*TN \cong p_1^*\xi + \epsilon_{M\times N}^1 + p_2^*TN \cong p_1^*\xi + p_2^*(TN + \epsilon_N^1).$$

By the assumption,

$$p_1^*\xi + p_2^*(TN + \epsilon_N^1) \cong p_1^*\xi + p_2^*(\epsilon_N^{n+1}).$$

Finally, we have the isomorphisms

$$p_1^*\xi + p_2^*(\epsilon_N^{n+1}) \cong p_1^*(\xi + \epsilon_M^{n+1}) \cong \epsilon_{M\times N}^{m+n}$$

which complete the proof. □

To provide the postponed proof of Lemma 4.2, we will need the following by A. Haefliger [3].

THEOREM [Haefliger]. *Assume V, X are smooth manifolds of respective dimensions n, k and assume V is compact. Suppose $2k \geq 3(n+1)$. Let $f : V \to X$ be a continuous map such that f is an embedding in a neighborhood of ∂X and $f(\partial V) \cap f(V - \partial V) = \phi$. Assume $\pi_i(f) = 0$ for $0 \leq i \leq 2n - k + 1$. Then f is homotopic to an embedding relative to a neighborhood of ∂V.*

Also we need the following fact for which we refer to a work by A. Hatcher [4]. (This must be well known, perhaps with a slightly different condition on the dimensions, even if the authors had problem with finding a more appropriate reference.) In the following, a concordance F between $f, g : M \to Q$ means a proper embedding $F : M \times I \to Q \times I$ such that $F(x, 0) = (f(x), 0)$ and $F(x, 1) = (g(x), 1)$ for any $x \in M$ and an isotopy means a homotopy through embeddings.

THEOREM [Hatcher]. *Let Q, M be smooth manifolds with respective dimensions q, m. Assume there is a concordance $F : M \times I \to Q \times I$ between two embeddings $f, g : M \to Q$ and $q - m \geq 3$, $q \geq 6$. Then f, g are isotopic to each other.*

Proof. According to A. Hatcher ([4]), in particular, Remark 3, p. 229 together with the second paragraph of §2), under the given condition, F is homotopic to the concordance $f \times 1 : M \times I \to Q \times I$ relative to $M \times \{0\}$ through concordances. Now restrict the homotopy at $M \times \{1\} \equiv M$ to obtain the isotopy from g to f. □

Proof of Lemma 4.2. Let M denote the manifold and $f, g : M \to \mathbb{C}^m$ be the two embeddings. Then since $\mathbb{C}^m$ is contractible there is a homotopy $H : M \times I \to \mathbb{C}^m$ from f to g. Let $\bar{H} : M \times I \to \mathbb{C}^m \times I$ denote the map defined by $\bar{H}(x, t) = (H(x, t), t)$ for any $(x, t) \in M \times I$.

We apply the above theorem by Haefliger to conclude that $\bar{H}$ is homotopic to a concordance $F : M \times I \to \mathbb{C}^n \times I$ rel $M \times \{0, 1\}$. Here a concordance means simply an embedding such that $F^{-1}(X \times \{0, 1\}) = M \times \{0, 1\}$. Note that, since M is simply connected and $\mathbb{C}^n$ is contractible, we have $\pi_i(\bar{H}) = \pi_i(f) = 0$ for $i = 0, 1, 2$ and $2(m+1) - (2m+1) + 1 = 2$. Also note that $2(2m+1) \geq 3(m+1+1)$ if $m \geq 4$.

However the concordance F implies the existence of an isotopy from f to g according to the above theorem by A. Hatcher since $2m - m \geq 3$ and $2m \geq 6$ for any $m \geq 4$. □

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