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ON STRONG LAWS OF LARGE NUMBERS FOR 2-DIMENSIONAL POSITIVELY DEPENDENT RANDOM VARIABLES

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ABSTRACT. In this paper we obtain strong laws of large numbers for 2-dimensional arrays of random variables which are either pairwise positive quadrant dependent or associated. Our results imply extensions of Etemadi's strong laws of large numbers for nonnegative random variables to the 2-dimensional case.

1. Introduction

In the last years there has been growing interest in concepts of positive dependence for families of random variables (see for example, Blocks and Ting (1981), Karlin and Rinott (1980), Shaked (1982) and the references therein). Such concepts are of considerable use in deriving inequalities in probability and statistics. Lehmann (1966) introduced the notion of positive quadrant dependence: A sequence $\{X_i : i \geq 1\}$ of random variables is said to be pairwise positive quadrant dependent (pairwise PQD) if for any real r_i, r_j and $i \neq j$ $P\{X_i > r_i, X_j > r_j\} \geq P\{X_i > r_i\}P\{X_j > r_j\}$. A much stronger concept than PQD was considered by Esary, Proschan, and Walkup (1967): A sequence $\{X_i : i \geq 1\}$ of random variables is said to be associated if for any finite collection $\{X_{j(1)}, \cdots, X_{j(n)}\}$ and any real coordinatewise nondecreasing functions

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f , g on R^n, $Cov(f\ (X_{j(1)},\cdots,X_{j(n)}\ )\ ,\ g(\ X_{j(1)},\cdots,X_{j(n)}\ )\ )\geq 0$, whenever the covariance is defined. Let us remark that associated random variables are always pairwise PQD and that (pairwise) independent random variables are (pairwise PQD) associated. For a sequence of random variables the strong law of large numbers are investigated extensively in the literature; Etemadi (1981) derived the strong law of large numbers for a sequence of pairwise independent, identically distributed random variables and extended it to the d-dimensional array of random variables. Etemadi (1983 a) studied the strong law of large numbers for a sequence of nonnegative random variables which are pairwise negatively dependent and Csőrgő , Tandori, and Totik (1983) investigated the strong laws of large numbers for sums of pairwise independent random variables. Birkel (1989) also studied the following strong laws of large numbers for sequences of random variables which are either pairwise positively quadrant dependent or associated:

THEOREM 1.1. (Birkel, 1989) *Let* $\{X_j\ :\ j\geq 1\}$ *be a sequence of pairwise PQD random variables with finite variance. Assume*

$$\text{(i)}\ \sum_{j=1}^{\infty}\sum_{i=1}^{j} j^{-2}\ Cov(X_i,X_j)<\infty,$$

$$\text{(ii)}\ \sup_{j\geq 1}\ E|X_j-EX_j|<\infty.$$

Then, as $n\to\infty$, $n^{-1}(S_n-ES_n)\to 0$ *a. s.*

THEOREM 1.2. (Birkel, 1989) *Let* $\{X_j\ :\ j\geq 1\}$ *be a sequence of associated random variables with finite variance. Assume that* (i) *of* Theorem 1.1 *holds. Then, as* $n\to\infty$, *we have* $n^{-1}(S_n-ES_n)\to 0$ *a. s.*

Let us remark that if the random variables are associated assumption (ii) in Theorem 1.1 is dropped. In this paper we extend Birkel's (1989) strong laws of large numbers for sums of random variables which are either pairwise PQD or associated (Theorems 1.1 and 1.2) to the 2-dimensional case. In Section 2 we obtain a strong law of large numbers for positively dependent nonnegative random variables by applying Theorem 1 in Etemadi (1983 b), a striking result without the independence hypothesis, and extend this result to the 2-dimensional case. In Section

3 we derive strong laws of large numbers for 2-dimensional arrays of random variables which are either pairwise positive quadrant dependent or associated.

Throughout this work $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ is a 2-dimensional array of random variables defined on a probability space (Ω, F, P), and let $S_{\underline{n}} = \sum_{\underline{1} \leq \underline{j} \leq \underline{n}} X_{\underline{j}} = \sum_{j_2=1}^{n_2} \sum_{j_1=1}^{n_1} X_{(j_1,j_2)}$, for $\underline{n} = (n_1, n_2)$ and $\underline{j} = (j_1, j_2)$. We shall use the following notations and conventions:
(i) $[a]$ will denote the integer part of the real number a, i.e., the greatest integer smaller than or equal to a,
(ii) $\underline{1} = (1, 1)$, $|\underline{j}| = j_1 \times j_2$ for $\underline{j} = (j_1, j_2)$,
(iii) $\underline{n} = (n_1, n_2) \to \infty$ means that $n_1 \times n_2 \to \infty$,
(iv) $X^+ = \max(0, X)$ and $X^- = \max(0, -X)$.

2. Preliminaries

From Theorem 1 of Etemadi (1983b) by putting $w_n = 1$ for $n \geq 1$ we obtain the following lemma for a sequence of positively dependent nonnegative random variables.

LEMMA 2.1. (Etemadi, 1983b) *Let $\{X_j : j \geq 1\}$ be a sequence of positively dependent nonnegative random variables with finite second moments. Assume*

$$\text{(i) } \sup_{j \geq 1} EX_j < \infty,$$

$$\text{(ii) } \sum_{j=1}^{\infty} \sum_{i=1}^{j} Cov(X_i, X_j)/j^2 < \infty.$$

Then as $n \to \infty$ $(S_n - ES_n)/n \to 0$ a. s.

Following theorem will play the key role to derive the main results.

THEOREM 2.1. *Let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of positively dependent nonnegative random variables with finite second* *moments. Assume*

$$\text{(i)}\ \sup_{\underline{j}\geq\underline{1}} EX_{\underline{j}} < \infty,$$

$$\text{(ii)}\ \sum_{\underline{j}\geq\underline{1}} \sum_{|\underline{j}|\geq|\underline{i}|\geq 1} Cov(X_{\underline{i}}, X_{\underline{j}})/|\underline{j}|^2 < \infty,$$

where $|\underline{j}| = j_1 \times j_2$. *Then as* $\underline{n} \to \infty, (S_{\underline{n}} - ES_{\underline{n}})/|\underline{n}| \to 0$ *a. s.*

Proof. Let $a > 1,\ b > 1$ and $n_{\underline{k}} = ([a^{k_1}], [b^{k_2}])$ for $\underline{k} = (k_1, k_2)$. By Chebyshev's inequality for every $\varepsilon > 0$

$$\begin{aligned}
&\sum_{\underline{k}\geq\underline{1}} P\{|S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}| \geq \epsilon|n_{\underline{k}}|\} \\
&\leq b_1 \sum_{\underline{k}\geq\underline{1}} Var(S_{n_{\underline{k}}}) \ / \ |n_{\underline{k}}|^2 \\
&\leq b_1 \sum_{\underline{k}\geq\underline{1}} \sum_{\underline{1}\leq\underline{j}\leq n_{\underline{k}}} \sum_{\underline{1}\leq\underline{i}\leq n_{\underline{k}}} Cov(X_{\underline{i}}, X_{\underline{j}}) \ / \ |n_{\underline{k}}|^2 \\
&\leq b_2 \sum_{\underline{k}\geq\underline{1}} \sum_{\underline{1}\leq\underline{j}\leq n_{\underline{k}}} \sum_{\underline{1}\leq\underline{i}\leq n_{\underline{k}}} Cov(X_{\underline{i}}, X_{\underline{j}})/|n_{\underline{k}}|^2 \qquad (2.1)\\
&= b_2 \sum_{\underline{k}\geq\underline{1}} \frac{1}{|n_{\underline{k}}|^2} \sum_{\underline{1}\leq\underline{j}\leq n_{\underline{k}}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}})/|n_{\underline{k}}|^2 \\
&= b_2 \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}}) \sum_{\{\underline{k}:n_{\underline{k}}\geq\underline{j}\}} \frac{1}{|n_{\underline{k}}|^2} \\
&\leq b_3 \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}}) \ / \ |\underline{j}|^2 < \infty.
\end{aligned}$$

The last inequality of (2.1) follows from the following: First note that

$$\sum_{\{\underline{k}:n_{\underline{k}\geq\underline{j}}\}} \frac{1}{|n_{\underline{k}}|^2} = \sum_{\underline{k}\geq\underline{k_0}} \frac{1}{|n_{\underline{k}}|^2} \qquad (2.2)$$

where $\underline{k_0} = \min\{\underline{k} : n_{\underline{k}} \geq \underline{j}\}$. Then the right hand side of (2.2) yields

$$\begin{aligned} \sum_{\underline{k} \geq \underline{k_0}} \frac{1}{|n_{\underline{k}}|^2} \leq C \sum_{\underline{k} \geq \underline{k_0}} \frac{1}{a^{2k_1} b^{2k_2}} &= C \sum_{k_2 \geq k_0''} \sum_{k_1 \geq k_0'} \frac{1}{a^{2k_1} b^{2k_2}} \\ &\leq \frac{D}{a^{2k_0'} b^{2k_0''}} \\ &\leq \frac{D}{|n_{\underline{k_0}}|^2} \leq \frac{E}{|\underline{j}|^2} \end{aligned}$$

where $\underline{k_0} = (k_0', k_2'')$ and C, D and E are some positive constants. Thus by the Borel-Cantelli lemma it follows from (2.1) that

$$(S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}) \ / \ |n_{\underline{k}}| \to 0 \ a.\ s. \tag{2.3}$$

Now given $\underline{k} = (k_1, k_2)$, positive integers k_1, k_2 for $n_{\underline{k}} \leq \underline{n} \leq n_{\underline{k}+\underline{1}}$ we have

$$\left| \frac{S_{\underline{n}} - ES_{\underline{n}}}{|\underline{n}|} \right| \leq \left| \frac{S_{n_{\underline{k}+\underline{1}}} - ES_{n_{\underline{k}+\underline{1}}}}{|n_{\underline{k}+\underline{1}}|} \right| \frac{|n_{\underline{k}+\underline{1}}|}{|n_{\underline{k}}|} + \frac{ES_{n_{\underline{k}+\underline{1}}} - ES_{n_{\underline{k}}}}{|n_{\underline{k}}|} \tag{2.4}$$

by the monotonicity of $S_{\underline{n}}$. Let $a > 1,\ b > 1$ and for each $n_{\underline{k}} = (n_{k_1},\ n_{k_2})$ set $n_{k_1} = [a^{k_1}], n_{k_2} = [b^{k_2}]$. Then from (i), (2.3) and (2.4) one can easily verify that

$$\lim\ \sup\ (|S_{\underline{n}} - ES_{\underline{n}}| \ / \ |\underline{n}|) \ \leq \sup_{\underline{j} \geq \underline{1}} EX_{\underline{j}}(ab - 1),$$

for every $a > 1$ and $b > 1$ which concludes the proof since both a and b may be arbitrary close to 1.

LEMMA 2.2. *Let X be a random variable with finite second moment. Define*

$$X^+ = \max(X, 0) \text{ and } X^- = \max(-X, 0).$$

Then

$$\begin{aligned} &\text{(i)} \quad Var(X^+) \leq Var(X), \\ &\text{(ii)} \quad Var(X^-) \leq Var(X). \end{aligned}$$

Proof. Note that for all nondecreasing (nonincreasing) functions f and g,

$$\text{(2.5)} \qquad Cov(f(X), g(X)) \geq 0,$$

whenever the covariance is defined. Applying (2.5) with $f(x) = x - x^+$, $g(x) = x^+$, and $f(x) = x - x^+$, $g(x) = x$, we obtain

$$\text{(2.6)} \qquad Var(X^+) \leq Cov(X, X^+), \; Cov(X, X^+) \leq Var(X).$$

Thus (2.6) proves (i). Since $X^- = (-X)^+$, (ii) follows from (i). □

LEMMA 2.3. (Birkel, 1989) *Let X_i and X_j be PQD. Then*

$$\text{(i)} \quad 0 \leq Cov(X_i^+, X_j^+) \leq Cov(X_i, X_j),$$
$$\text{(ii)} \quad 0 \leq Cov(X_i^-, X_j^-) \leq Cov(X_i, X_j).$$

Proof. See the proof of Theorem in Birkel (1989). □

Following corollary is an extension of Corollary 1 of Etemadi (1983a) to the 2-dimensional case.

COROLLARY 2.1. *Let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of pairwise independent random variables with finite second moments. Assume*

$$\text{(i)} \quad \sup_{\underline{j} \geq \underline{1}} E|X_{\underline{j}} - EX_{\underline{j}}| < \infty,$$
$$\text{(ii)} \quad \sum_{\underline{j} \geq \underline{1}} |\underline{j}|^{-2} Var(X_{\underline{j}}) < \infty.$$

Then as $\underline{n} \to \infty$, $(S_{\underline{n}} - ES_{\underline{n}})/|\underline{n}| \to$ a. s.

Proof. Use the arguments in the proof of Corollary 1 of Etemadi (1983a). Consider the 2-dimensional array of random variables $\{(X_{\underline{j}} - EX_{\underline{j}})^+ : \underline{j} \geq \underline{1}\}$ and its corresponding sum, say $S^*_{\underline{n}}$. According to Lemma 2.2 $Var(X_{\underline{j}} - EX_{\underline{j}})^+ \geq Var(X_{\underline{j}} - EX_{\underline{j}}) = Var(X_{\underline{j}})$ for $\underline{j} \geq \underline{1}$ and $Cov(X_i^+, X_j^+) = 0$ for $i \neq j$ according to Lemma 2.3. Thus we clearly have, as $\underline{n} \to \infty$, $|\underline{n}|^{-1}(S^*_{\underline{n}} - ES^*_{\underline{n}}) \to 0$ *a. s.* by using Theorem 2.1. A similar consideration for negative part, say $S^{**}_{\underline{n}}$, together with the fact that $ES^*_{\underline{n}} - ES^{**}_{\underline{n}} = 0$ completes the proof of Corollary 2.1. □

3. Main results

The following theorem is the strong law of large numbers for 2-dimensional pairwise PQD random variables.

THEOREM 3.1. *Let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of pairwise PQD random variables with finite variance. Assume*

$$\text{(i)}\quad \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}})/|\underline{j}|^2 < \infty,$$

$$\text{(ii)}\quad \sup_{\underline{j}\geq\underline{1}} E|X_{\underline{j}} - EX_{\underline{j}}| < \infty.$$

Then, as $\underline{n} \to \infty$, $|\underline{n}|^{-1}(S_{\underline{n}} - ES_{\underline{n}}) \to 0$ a. s.

Proof. First note that $(X_{\underline{i}}, X_{\underline{j}})$ is PQD if and only if

$$Cov(f(X_{\underline{i}}), g(X_{\underline{j}})) \geq 0,$$

for all nondecreasing (nonincreasing) functions f, g such that the covariance exists (see Lemma 1 of Lehmann (1966)). Hence $X_{\underline{j}} - EX_{\underline{j}}$, $\underline{j} \geq \underline{1}$, are pairwise PQD and without loss of generality, we may assume that, for $\underline{j} \geq \underline{1}$, $EX_{\underline{j}} = 0$. Now we consider $\{X_{\underline{j}}^{+} : \underline{j} \geq \underline{1}\}$ and its corresponding sum, say $S_{\underline{n}}^{*} = \sum_{\underline{1}\leq\underline{i}\leq\underline{n}} X_{\underline{i}}^{+}$. Our assumptions (i) and (ii) together with Theorem 2.1 and Lemma 2.3 imply that, as $n \to \infty$, $|\underline{n}|^{-1}(S_{\underline{n}}^{*} - ES_{\underline{n}}^{*}) \to 0$ *a. s.* A similar consideration for the negative parts, say $S_{\underline{n}}^{**} = \sum_{\underline{1}\leq\underline{i}\leq\underline{n}} X_{\underline{i}}^{-}$, and fact that $ES_{\underline{n}}^{*} - ES_{\underline{n}}^{**} = 0$ complete the proof of Theorem 3.1. □

REMARK. For the one dimensional random variables an example of Csörgő, Tandori and Totik (1983) shows that even for pairwise independent (and hence pairwise PQD) random variables condition (i) alone does not imply the strong law of large numbers (see [1], [3]).

Newmann and Wright (1982) introduced the following maximal inequality for 2-dimensional associated random variables.

LEMMA 3.1. (Newman, Wright 1982) *Let $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ be a 2-dimensional array of associated variables with $EX_{\underline{j}} = 0, EX_{\underline{j}}^2 < \infty$.*

For $\lambda_2 > \lambda_1 \geq 0$, *we have*

$$(3.1)\quad P\{\max_{\underline{1}\leq \underline{j}\leq \underline{n}} S_{\underline{j}} \geq \lambda_2\} \leq 3^{\frac{3}{2}}2^{-1}(ES_{\underline{n}}^2/(\lambda_2-\lambda_1)^2)^{\frac{3}{4}}[P(S_{\underline{n}} \geq \lambda_1)]^{\frac{1}{4}}.$$

Proof. See the proof of Theorem 10 of Newman and Wright (1982). □

If the random variables are associated, assumption (ii) in Theorem 3.1 may be dropped and we need maximal inequality to prove the following theorem:

THEOREM 3.2. *Let* $\{X_{\underline{j}} : \underline{j} \geq \underline{1}\}$ *be a 2-dimensional array of associated random variables with finite variance. Assume*

$$\text{(i)} \sum_{\underline{j}\geq \underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}}) \;/\; |\underline{j}|^2 < \infty.$$

Then, as $\underline{n} \to \infty$, $|\underline{n}|^{-1}(S_{\underline{n}} - ES_{\underline{n}}) \to 0$ *a. s.*

Proof. Since the random variables $X_{\underline{j}} - EX_{\underline{j}}$, $\underline{j} \geq \underline{1}$ are associated by (P_4) of Esary, Proschan and Walkup (1967), without loss of generality we may assume that, for $\underline{j} \geq \underline{1}$, $EX_{\underline{j}} = 0$. Now use the ideas in the proofs of Theorems 2.1 and 3.1. Then by Chebyshev's inequality for every $\varepsilon > 0$,

$$(3.2)\quad \begin{aligned} &\sum_{\underline{k}\geq \underline{1}} P\{|S_{n_{\underline{k}}} - ES_{n_{\underline{k}}}| \;/\; |n_{\underline{k}}| > \epsilon\} \\ &\geq c\sum_{\underline{k}\geq \underline{1}} Var(S_{n_{\underline{k}}}) \;/\; |n_{\underline{k}}|^2 \\ &\geq c\sum_{\underline{j}\geq \underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}}) \;/\; |\underline{j}|^2 < \infty, \end{aligned}$$

where c is an unimportant constant. Thus by the Borel-Cantelli lemma, as $n_{\underline{k}} \to \infty$,

$$(3.3)\quad (S_{n_{\underline{k}}} - ES_{n_{\underline{k}}})/|n_{\underline{k}}| \to 0 \text{ a. s.}$$

By standard argument, it suffices to show that, as $n_{\underline{k}} \to \infty$,

$$(3.4)\quad |n_{\underline{k}}|^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}+\underline{1}}} |S_{\underline{i}} - S_{n_{\underline{k}}}| \to 0 \text{ a. s.}$$

Let $a > 1$, $b > 1$ and $n_{\underline{k}} = (n_{k_1}, n_{k_2}) = ([a]^{k_1}, [b]^{k_2})$ and $\underline{k} = (k_1, k_2)$ and let $\lambda_2 = \epsilon|n_{\underline{k}}|$ and $\lambda_1 = \frac{1}{2}\varepsilon|n_{\underline{k}}|$. Using Chebyshev's inequality and applying (3.1) of Lemma 3.1 we have

(3.5)

$$\begin{aligned} P & \ \{|n_{\underline{k}}|^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}+\underline{1}}} (S_{\underline{i}} - S_{n_{\underline{k}}}) \geq \epsilon\} \\ \leq & \ 3^{\frac{3}{2}} 2^{-1} \Big(E(S_{n_{\underline{k}+1}} - S_{n_{\underline{k}}})^2 \ / \ (\tfrac{1}{2}\epsilon|n_{\underline{k}}|)^2 \Big)^{\frac{3}{4}} \Big(E(S_{n_{\underline{k}+\underline{1}}} - S_{n_{\underline{k}}})^2 \ / \ (\tfrac{1}{2}\epsilon|n_{\underline{k}}|)^2 \Big)^{\frac{1}{4}} \\ \leq & \ c|n_{\underline{k}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}} - S_{n_{\underline{k}}}) \\ \leq & \ c|n_{\underline{k}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}}) \\ \leq & \ a^2b^2c|n_{\underline{k}+1}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}}) \end{aligned}$$

since the $X_{\underline{j}}$ are nonnegatively correlated. Replacing the random variables $X_{\underline{n}}$ by their negatives (which are also associated according to (P_4) of Esary, Proschan and Walkup (1967)) we get the analogous inequality

(3.6) $$P\{|n_{\underline{k}}|^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}+\underline{1}}} -(S_{\underline{i}} - S_{n_{\underline{k}}}) \geq \epsilon\} \leq a^2b^2c|n_{\underline{k}+\underline{1}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}})$$

(3.5) and (3.6) imply

(3.7) $$P\{|n_{\underline{k}}|^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}+\underline{1}}} |S_{\underline{i}} - S_{n_{\underline{k}}}| \geq \epsilon\} \leq 2a^2b^2c|n_{\underline{k}+\underline{1}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}})$$

and hence

$$\begin{aligned} & \sum_{\underline{k}\geq\underline{1}} P\{|n_{\underline{k}}|^{-1} \max_{n_{\underline{k}}<\underline{i}\leq n_{\underline{k}+\underline{1}}} |S_{\underline{i}} - S_{n_{\underline{k}}}| \geq \epsilon\} \\ & \leq 2a^2b^2c \sum_{\underline{k}\leq\underline{1}} |n_{\underline{k}+\underline{1}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}}) \\ & \leq 2a^2b^2c \sum_{\underline{k}\geq\underline{1}} |n_{\underline{k}}|^{-2} Var(S_{n_{\underline{k}+\underline{1}}}) \\ & \leq 2a^2b^2c \sum_{\underline{j}\geq\underline{1}} \sum_{1\leq|\underline{i}|\leq|\underline{j}|} Cov(X_{\underline{i}}, X_{\underline{j}}) \ / \ |\underline{j}|^2 \\ & < \infty \end{aligned}$$

according to the above consideration (3.2). Again applying the Borel-Cantelli lemma, we obtain (3.4) which completes the proof of Theorem 3.2.

REMARK. The problem of extending (3.1) of Lemma 3.1 [Theorem 10 (formula (39)) of Newman and Wright (1982)] to d-dimensional case ($d > 2$) is presently an open question (see [9]). Therefore an extension of (3.5) to the general dimensional associated random variables with $d > 2$ is also open.

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