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PERMANENTS OF PRIME BOOLEAN MATRICES

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ABSTRACT. We study the permanent set of the prime Boolean matrices in the semigroup of Boolean matrices. We define a class M_n of prime matrices, and find all the possible permanents of the elements in M_n.

1. Introduction

Let $\mathcal{B} = \{0, 1\}$ be the Boolean algebra with operations $(+, \cdot)$ and the standard order $\leq$: $a + b = max\{a, b\}$ and $a \cdot b = min\{a, b\}$. Then the set B_n of all $n \times n$ matrices over $\mathcal{B}$ (Boolean matrices) forms a partially ordered multiplicative matrix semigroup under these Boolean operations and order.

For $m \times n$ Boolean matrices A and B, if the (i, j)-*th* entry A_{ij} of A is less than or equal to B_{ij} for each i and j, then we say B dominates A (or B contains A), and is denoted by $A \leq B$. If A dominates a permutation matrix, then A is called a Hall matrix and the set H_n of all $n \times n$ Hall matrices is a subsemigroup of B_n.

DEFINITION 1.1. Let G be a multiplicative semigroup with an identity element. A non-zero non-invertible element $A \in G$ is a prime element of G if A cannot be expressed as a product of two non-invertible elements of G. A is called factorizable in G if A is not prime in G.

For a Boolean matrix $A \in B_n$, A_{i*} and A_{*j} denote respectively the i-*th* row and the j-*th* column of A. The Boolean rank of A is the smallest integer r such that $A = B \cdot C$, where B and C are $n \times r$ and $r \times n$ Boolean matrices respectively (the Boolean rank of any zero matrix is zero). A

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is called a rank-one matrix if A can be expressed as a Boolean product of a length n column vector and a row vector. Note that $A = B \cdot C$ implies $A = \sum_{i=1}^{r} R_i$, where $R_i = B_{*i} \cdot C_{i*}$. Therefore the Boolean rank of A is the minimum number of rank-one dominated submatrices of A whose Boolean sum is A.

A row A_{k*} of $A \in B_n$ is an independent row of A if A_{k*} cannot be expressed as a Boolean sum of some other rows of A (independent column of A is defined similarly). Then the row (respectively column) rank of A is the maximum number of independent rows (respectively columns) of A. A row or a column of a matrix is called a line, and the term rank of A is the minimum number of lines needed to cover all the nonzero entries of A. We say a matrix A has a line domination property if a row (or a column) of A dominates another row (column).

PROPOSITION 1.2 (D. de Caen and Gregory [4]). *Let $A \in B_n$ be a prime Boolean matrix. Then*

(i) *The Boolean rank of A is n.*
(ii) *A does not have a line domination property.*

It follows from *König's Theorem* that the term rank of $A \in B_n$ is greater than or equal to its Boolean rank. Thus if the Boolean rank of A is n, then the term rank of A is also n. Therefore a prime Boolean matrix is a Hall matrix. Also a prime Boolean matrix has full row rank and full column rank since the Boolean rank of $A \in B_n$ is greater than or equal to the row rank and the column rank of A. Also note that the above statements (i) and (ii) are logically independent [6, 7].

DEFINITION 1.3. For an $n \times n$ Boolean matrix $A \in B_n$, the permanent $per(A)$ of A is the number of $n \times n$ permutation matrices dominated by A.

For $A \in B_n$, $per(A)$ is equal to the real sum of $\sum_{\sigma \in S_n} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{n\sigma(n)}$, where S_n denotes the set of all permutations on $\{1, ..., n\}$.

$A \in B_n$ is a fully indecomposable matrix if A is not permutation equivalent to a matrix of the form $\begin{pmatrix} B_1 & * \\ \mathbf{O} & B_2 \end{pmatrix}$, where B_1 and B_2 are square matrices and $\mathbf{O}$ denotes a zero matrix. Also A is partly decomposable if A is not fully indecomposable, and A is nearly decomposable if deleting any positive entry of A results in a partly decomposable matrix. For $A \in B_n$, $\sigma(A)$ denotes the number of nonzero entries in A.

PROPOSITION 1.4. *Let $A \in B_n$ be a prime Boolean matrix. Then,*

(i) *A is permutation equivalent to a direct sum of a fully indecomposable prime and an identity matrix or a fully indecomposable matrix.*

(ii) *$per(A) \geq 2$*

Proof. (i) Refer to Cho [5]. (ii) It is well known that for an $n \times n$ fully indecomposable matrix B, $per(B) \geq \sigma(B) - 2n + 2$ (Minc's inequality), and thus $per(B) \geq 2$. Therefore $per(A) \geq 2$ by (i) and Minc's inequality when $A \in B_n$ is a prime Boolean matrix. □

It is well known that any nonsingular real matrix in the semigroup R_n of $n \times n$ real matrices can be written as a product of elementary matrices. Similarly every $n \times n$ Boolean matrix of Boolean rank n can be expressed as a product of prime matrices and elementary matrices [5]. Some other properties of the prime matrices are given in [1, 2, 8, 9].

2. Permanent set of prime Boolean matrices

Let P_n denote the set of all prime matrices in B_n, and $Q_n = \{per(A) | A \in P_n\}$. Then what is the structure of the permanent set Q_n and what is the maximum value of Q_n? Is there any gap in Q_n? To give a partial answer, we define a subset M_n of P_n such that $M_n = \{A | A \in P_n$ and $per(A) = \sigma(A) - 2n + 2\}$, and we study the permanent set $N_n = \{per(A) | A \in M_n\}$ in this section.

PROPOSITION 2.1 (Brualdi and Gibson [3]). *A fully indecomposable matrix $A \in B_n$ has $per(A) = \sigma(A) - 2n + 2$ if and only if there exists an integer p with $0 \leq p \leq n-1$ such that A is permutation equivalent to a matrix $N = \begin{pmatrix} G & \mathbf{O} \\ F & H \end{pmatrix}$, where F is an $(n-p) \times (p+1)$ matrix and G and H^T are matrices with exactly two 1's in each row.*

For $A \in B_n$, let $\sigma_r(A)$ (respectively, $\sigma_c(A)$) denote the maximum row (column) sum of the rows (columns) of A. A rank-one dominated submatrix R of A is nontrivial if $\sigma_r(R) \geq 2$ and $\sigma_c(R) \geq 2$. We can define $A - R$ as usual when $R \leq A$. $E_n(i,j)$ denotes an $n \times n$ Boolean matrix whose (i,j)-*th* entry is the only nonzero entry.

LEMMA 2.2. *Let the permanent of $A \in B_n$ be $\sigma(A) - 2n + 2$, and let A have no line domination property. Then, A is in M_n if and only if A is fully indecomposable and $A - R$ has no $(n-p) \times (p+1)$ zero submatrix for each nontrivial rank-one dominated submatrix R of A.*

Proof. By Proposition 2.1, if A is fully indecomposable, then A is permutation equivalent to a matrix $N = \begin{pmatrix} G & O \\ F & H \end{pmatrix}$, where F is an $(n-p) \times (p+1)$ matrix and G and H^T have exactly two 1's in each row. Now let $D = \begin{pmatrix} G & O \\ O & H \end{pmatrix}$ and $M = \begin{pmatrix} O & O \\ F & O \end{pmatrix}$, and let $E = D + E_n(\alpha, \beta)$, where $E_n(\alpha,\beta) \leq M$. Note that the term rank of E is n since there exists a permutation matrix P in N dominating $E_n(\alpha,\beta)$. Also note that if there is a nontrivial rank-one dominated submatrix R of N, then we have $R \leq M$ since N has no line domination property.

(Only if part) By proposition 1.4 and Minc's inequality, A is fully indecomposable since A is in M_n. Thus without loss of generality, we may assume that A is of the form N. Now suppose that there is an $(n-p) \times (p+1)$ zero submatrix of $N - R$ for some nontrivial rank-one dominated submatrix R of N. Then there are $n-1$ many rows and columns of $N - R$ such that the sum of these lines of $N - R$ and R is N. Thus we can have n many rank-one dominated submatrices of A such that the sum is A and one of them is a nontrivial rank-one matrix. Therefore A is a factorizable matrix, a contradiction. Thus for each nontrivial rank-one dominated submatrix R of A, there is no $(n-p) \times (p+1)$ zero submatrix of $A - R$.

(If part) By the assumption, we may assume that A is of the form N since A is fully indecomposable. Now let $N = B \cdot C$ for some B and C in B_n with $R_i = B_{*i} \cdot C_{i*}$. Then we claim that B or C is a permutation matrix, and argue as follows: If all the R_i's are trivial rank-one dominated submatrices of N, then each nonzero line (row or column) of R_i is contained in a line of N. Therefore the positive entries of N can be covered by n-many lines of N and N is a partly decomposable matrix, a contradiction. So one of R_i's, say R_j, is a nontrivial rank-one dominated submatrix of N. Then $R_j \leq M$ and R_j must contain an $E_n(\alpha, \beta)$ since we assume that $N - R_j$ does not have $(n-p) \times (p+1)$ zero submatrix. Since the term ranks of D and E are $n-1$ and n respectively, we need at least n many rows and columns of N to cover the positive entries of $N - R_j$. Hence N and A are primes in B_n since every nontrivial rank-one dominated submatrix of N is contained in M. □

THEOREM 2.3. *Let $n \geq 3$ and $N_n = \{per(A) | A \in M_n\}$. Then,*

(i) *The maximum ρ_n of the set N_n is $\left[\frac{n^2-2n+5}{4}\right]$, and the minimum of N_n is 2.*

(ii) *For each s with $2 \leq s \leq \rho_n$, there exists a prime matrix $A \in M_n$ such that $per(A) = s$.*

Proof. When $n = 3$ or $n = 4$, the theorem holds since every nearly decomposable matrix is in M_n [5]. Thus we consider the case when $n \geq 5$ in the following proof.

(i) Let $A \in M_n$. By Proposition 2.1 and Lemma 2.2, we may assume that A is of the form $N = \begin{pmatrix} G & O \\ F & H \end{pmatrix}$, where F is an $(n-p) \times (p+1)$ matrix and G and H^T have exactly two 1's in each row. Let r and t be the number of minimum number of zero entries in the rows and the columns of F respectively. Note that r and t cannot be 0 since there is no all-one row and column in F.

Now we claim that $per(N) \leq (n-p-1)p + 1$, and argue as follows: First, if both r and s are greater than one (so each row and column of F contains at least two 0's), then the claim holds by the counting argument. Second, let r be 1. Also let $\sigma(F_{k*}) = p$ for some k and the (i,j)-*th* entry of N contained in F_{k*} be 0. Then for each α, the (α, j)-*th* entry of N located in $G_{\alpha *}$ should be 1 not to be contained in the i-*th* row N_{i*} of N. So G is permutation equivalent to the following Boolean matrix

$$\mathcal{G} = \begin{pmatrix} 1 & 1 & & & \\ 1 & & 1 & & \\ \vdots & & & \ddots & \\ 1 & & & & 1 \end{pmatrix},$$

where the unspecified entries are all zero. Now consider the case (1): for some β, the (β, j)-*th* entry of N located in F is 1, and the case (2): there is no such β satisfying the condition specified in the case (1).

In case (1), except the (β, j)-*th* entry all the entries of $N_{\beta *}$ located inside of F are zero, and this means the claim is true. In case (2), we have a zero column F_{*j}. Since each column of F must have at least one zero entry, we see the claim is true by the counting argument. From all these cases, we have $per(N) = \sigma(F) \leq (n-p-1)p+1$. Let Ω be the matrix N whose blocks F, G, and H are of the form $\mathcal{F}$, $\mathcal{G}$, and $\mathcal{H}$ respectively, and $\mathcal{F}$, $\mathcal{H}$ are as follows:

$$\mathcal{F} = \begin{pmatrix} 0 & & & & \\ \vdots & & & * & \\ 0 & & & & \\ 0 & & & & \\ 1 & 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad \mathcal{H} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \\ 1 & 1 & \cdots & 1 \end{pmatrix},$$

where the entries in the *-part of $\mathcal{F}$ are all one and the unspecified entries of $\mathcal{H}$ are all zero. Then the permanent of such matrix Ω is exactly equal to $(n-p-1)p+1$ and $\Omega \in M_n$ by Lemma 2.2. Thus the maximum possible permanent of the matrices in M_n having such an $(n-p) \times (p+1)$ submatrix $\mathcal{F}$ is $(n-p-1)p+1$. Since the maximum of the p-variable quadratic function $-p^2+(n-1)p+1$ is taken when $p = \frac{n-1}{2}$, the maximum possible permanent ρ_n of the matrices in M_n is $\frac{n^2-2n+4}{4}$ for even n and $\frac{n^2-2n+5}{4}$ for odd n. Note that $per(A) \geq 2$ for $A \in M_n$ by Proposition 1.4.

(ii) For the low-left block $\mathcal{F}$ of the matrix Ω considered in (i), let $\mathbf{f}_r$ be a matrix obtained from $\mathcal{F}$ by replacing r many nonzero entries in the $*-$part of $\mathcal{F}$ by zero such that $\mathbf{f}_r$ has no zero row and no zero column. Now let Ω_r be the matrix obtained from Ω by replacing $\mathcal{F}$ by $\mathbf{f}_r$. Note that $per(\Omega_r) = (n-p-1)p+1-r$. Also Ω_r is fully indecomposable and prime by Lemma 2.2, and thus $\Omega_r \in M_n$. Using this method, we can construct a prime matrix A in M_n with $per(A) = s$ for any s with $max\{n-p, p+1\} \leq s \leq (n-p-1)p+1$.

Next, consider a prime matrix $\Lambda = \begin{pmatrix} G & \mathbf{O} \\ F & H \end{pmatrix}$, where F, G, and H are as follows:

$$F = \begin{pmatrix} 0 & \cdots & 0 & 1 \\ \vdots & & & 0 \\ 0 & & & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, G = \begin{pmatrix} 1 & 1 & & & \\ & 1 & \ddots & & \\ & & \ddots & 1 & \\ & & & 1 & 1 \end{pmatrix}, H = \begin{pmatrix} 1 & & & \\ 1 & \ddots & & \\ & \ddots & \ddots & 1 \\ & & & 1 \end{pmatrix},$$

where the unspecified entries of the matrices are all zero. Let $\mathbf{g}_r$ be a matrix obtained from F by replacing r many zero entries of F by one such that there is no line domination in $\mathbf{g}_r$. Now let Λ_r be the matrix obtained from Λ by replacing F by $\mathbf{g}_r$. Then by Lemma 2.2 $\Lambda_r \in M_n$ and $per(\Lambda_r) = 2 + r$. Note that there is no prime matrix strictly contained in Λ [5]. Thus we can construct a prime matrix A in M_n with $per(A) = s$ for any s with $2 \leq s \leq max\{n-p, p+1\} - 1$.

From these two kinds of constructions, we see that for each s with $2 \leq s \leq \rho_n$ there is a prime matrix A in M_n with $per(A) = s$. □

3. Closing remarks

Let P_n be the poset of prime matrices of B_n. For $A \in P_n$, A is called a minimal prime matrix if there is no prime matrix strictly dominated by A, and A is called a maximal prime matrix if there is no prime matrix containing A strictly. A is called a nearly factorizable matrix if A is prime and deleting any positive entry of A results in a factorizable matrix.

Consider the following 5 by 5 Boolean matrix

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Note that A is a fully indecomposable prime matrix. Also note that A is nearly factorizable since deleting any positive (i,j)-th entry of A results in a line domination in A. We can check that A is not a minimal prime matrix but a maximal prime matrix in B_5. In summary, there is a nearly factorizable matrix A in B_n such that A is not a minimal prime matrix but a maximal prime matrix in B_n. It seems to be interesting to study minimal primes and maximal primes.

Consider $Q_5 = \{per(A)|A \in P_5\}$, $M_5 = \{per(A)|A \in P_5,\ per(A) = \sigma(A) - 2\cdot 5 + 2\}$. We can check that there are six fully indecomposable prime matrices in B_5, and $Q_5 = \{2,3,4,5\}$. Also we can check that M_5 is the set of all fully indecomposable prime matrices of B_5, and $N_5 = \{2,3,4,5\}$. Thus the maximum of Q_5 is 5 and there is no gap between 2 and this maximum value 5. Once again we propose the following problems. (1) : Determine the maximum of Q_n. (2) : Determine the gaps (if any) in the set Q_n. In fact, the question (1) is the problem of finding the maximal primes in B_n.

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