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L^p SMOOTHNESS ON TRIEBEL-LIZORKIN SPACES IN TERMS OF SHARP MAXIMAL FUNCTIONS

Dedicated to Professor Jong-Uk Ham in celebration of his 60th birthday and in prayer for his recovery

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ABSTRACT. Taking into account the smoothness determined by sharp maximal functions, Sobolev type embedding results on Triebel-Lizorkin and Besov spaces are obtained.

1. Introduction

The primary purpose of this paper is to study the extent of smoothness determined by the condition $f_\alpha^\sharp \in L^p(\mathbb{R}^n)$ in which $f_\alpha^\sharp$ stands for the sharp maximal function, associated to a locally integrable function f on $\mathbb{R}^n$ and $0 \le \alpha \le 1$, defined by the formula

$$f_\alpha^\sharp(x) = \sup_{t>0} t^{-n-\alpha} \int_{B(x,t)} |\, f(y) - m_t(x) \,|\, dy\,, \tag{1-1}$$

where $B(x,t)$ denotes the open ball of radius $t>0$ with center at $x \in \mathbb{R}^n$ and $m_t(x)$ the average of f over $B(x,t)$.

The maximal operators (1-1) are introduced by A. P. Calderón and R. Scott [3] in their work on extending Sobolev type inequalities to $L^p + L^q$ spaces.

Regarding the degree of smoothness determined by α, it is well-known that $f_0^\sharp = f^\sharp$, often referred to as the sharp function in the

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theory of BMO space, and $\|f\|_{\mathrm{BMO}} = \|f^\sharp\|_\infty$. In the case of $\alpha = 1$, it is shown in the paper [3] that $\|f_1^\sharp\|_p \sim \|\nabla f\|_p$ (1) for $1 < p \le \infty$ and Coifman *et. al.* [6] recently extended it further to H^p spaces, $\|f_1^\sharp\|_p \sim \|\nabla f\|_{H^p}$ for $n/(n+1) < p \le 1$. When $0 < \alpha < 1$, it is plain that

(1-2)

$$\|f_\alpha^\sharp\|_p \le C_p \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|f(x)-f(y)|^p}{|x-y|^{n+\alpha p}}\,dx\,dy\right)^{1/p}, \quad 1 \le p < \infty,$$

the right side of which defines the norm for the homogeneous Sobolev space $\dot{W}^{\alpha,p}$.

As to the embedding question in terms of the maximal functions (1-1), Calderón and Scott proved the pointwise inequality

$$f^\sharp(x) \le C_\alpha I_\alpha(f_\alpha^\sharp)(x), \quad 0 < \alpha < 1, \tag{1-3}$$

where I_α denotes the Riesz potential of order α (also valid for $\alpha = 1$ if $n \ge 2$). Combined with the celebrated inequality of C. Fefferman and E. M. Stein [7]

$$\|f\|_p \le C_p \|f^\sharp\|_p, \quad 1 < p < \infty, \tag{1-4}$$

the inequality (1-3) gives sharper versions of Sobolev theorems on account of the known facts about Riesz potentials. However, we notice that the resulting embedding theorem is valid only in the range of $1 < p < n/\alpha$ and it appears that the other cases are not completely known even on L^p spaces.

In the present article, we set forth Triebel-Lizorkin and Besov spaces as the appropriate target spaces in consideration of the following reasons : First, those spaces generalize most of classical function spaces including L^p, H^p, BMO and Lipschitz spaces. Second, the current trend

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(1) Throughout this paper, this means that the two norms are equivalent:

$$C_1\|\nabla f\|_p \le \|f_1^\sharp\|_p \le C_2\|\nabla f\|_p.$$

shows that those are becoming more important in the theory of partial differential equations. Third, methodologically, those function spaces are definable in terms of certain dyadic maximal functions involving convolution family $\{ f * \varphi_k \}_{k\in\mathbb{Z}}$ and it is relatively easy to majorize them by $f_\alpha^\sharp$ via moment cancellations alluded to φ.

Our results will contain the complete answer to L^p embedding question in the full admissible range $n/(n+\alpha) < p \le \infty$. In addition, we obtain an extension and another proof of the inequality (1-4).

2. Inequalities on Triebel-Lizorkin and Besov Spaces

In accordance with Peetre [12], we begin with setting the following notations. For any Schwartz function ϕ on $\mathbb{R}^n$ such that

$$\operatorname{supp}(\hat{\phi}) \subset \{ 1/2 \le |\xi| \le 2 \} \quad \text{and} \quad |\hat{\phi}(\xi)| \ge c > 0 \quad \text{if} \quad 3/5 \le |\xi| \le 5/3$$

we put $\phi_\nu(x) = 2^{\nu n}\phi(2^\nu x)$ for each integer ν.[(2)] For each tempered distribution f on $\mathbb{R}^n$ and $\beta \in \mathbb{R}$, $\lambda > 0$, let

$$\begin{aligned} \phi_\nu^{**} f(x) &= \sup_{y\in\mathbb{R}^n} 2^{\nu\beta} \,|\, \phi_\nu * f(x-y) \,|\, (1 + 2^\nu |y|)^{-\lambda}, \\ \phi^{**} f(x) &= \left(\sum_{\nu\in\mathbb{Z}} |\phi_\nu^{**} f(x)|^q \right)^{1/q}, \quad 0 < q < \infty \end{aligned} \tag{2-1}$$

with the obvious ℓ^∞ norm for $q = \infty$.

The homogeneous Triebel-Lizorkin spaces $\dot{F}_p^{\beta,q}$ and the Besov spaces $\dot{B}_p^{\beta,q}$ are the spaces of tempered distributions with the finite quasi-norms

$$\begin{aligned} \|f\|_{\dot{F}_p^{\beta,q}} &= \|\, \phi^{**} f \,\|_p \quad \text{for} \quad \lambda > \frac{n}{\min(p,q)}, \\ \|f\|_{\dot{B}_p^{\beta,q}} &= \left(\sum_{\nu\in\mathbb{Z}} \|\phi_\nu^{**} f\|_p^q \right)^{1/q} \quad \text{for} \quad \lambda > \frac{n}{p}, \end{aligned} \tag{2-2}$$

(2) Note that ϕ itself can not have a compact support due to a version of Paley-Wiener theorem.

where $0 < p < \infty$, $0 < q \leq \infty$ and $\beta \in \mathbb{R}$ (Once again, the case of $q = \infty$ must be interpreted as the ℓ^∞ norm). In the limiting case of $p = \infty$, the $\phi_\nu^{**} f(x)$ in (2-1) are replaced by non-maximal versions $2^{\nu\beta} \,|\, \phi_\nu * f(x) \,|$ and the other formulations remain unchanged.

To state some of standard identifications of Triebel-Lizorkin spaces,

$$\dot{F}_p^{0,2} \simeq L^p, \quad 1 < p < \infty,$$
$$\dot{F}_p^{0,2} \simeq H^p, \quad 0 < p \leq 1,$$
$$\dot{F}_\infty^{0,2} \simeq \mathrm{BMO}, \quad \dot{F}_\infty^{\beta,\infty} \simeq \dot{\Lambda}_\beta, \quad \beta > 0, \tag{2-3}$$

which are obtainable by Littlewood-Paley theory. In general, $\dot{B}_p^{\beta,q}$ coincides with the generalized Lipschitz spaces (see [13] for instance). There are various duality theorems such as $\left(\dot{F}_p^{\beta,q}\right)^* \simeq \dot{F}_{p'}^{-\beta,q'}$ for $1 \leq p, q \leq \infty$, where p', q' denote the Hölder conjugate exponents of p, q.

The main theorem of this section reads as follows.

THEOREM A. *Let α, β be real numbers satisfying $0 \leq \alpha \leq 1$, $\alpha > \beta$. Suppose that $\left|\{\,|f| > \epsilon\,\}\right| < \infty$ for every $\epsilon > 0$ and $f_\alpha^\sharp \in L^p$ for $n/(n+\alpha) < p < n/(\alpha - \beta)$. Then for each $0 < q \leq \infty$,*

$$\|f\|_{\dot{F}_r^{\beta,q}} \leq C\,\|f_\alpha^\sharp\|_p, \quad 1/r = 1/p - (\alpha - \beta)/n. \tag{2-4}$$

Indeed, with the above exponents p, r and reals α, β, if $0 < q \leq 1$,

(2-5)

$$\phi^{**} f(x) \leq C\,\|f_\alpha^\sharp\|_p^{1-p/r} \left[f_\alpha^\sharp(x) \right]^{p/r}, \quad \lambda > \max\,(n + \alpha,\, n/q)\,.$$

REMARK 1. The condition that $\left|\{\,|f| > \epsilon\,\}\right| < \infty$ for every $\epsilon > 0$ is clearly necessary. It is plain to observe that $f_\alpha^\sharp \in L^p$ only if $p > n/(n+\alpha)$ under this condition.

In order to prove the theorem, we shall need a chain of lemmas.

LEMMA 2.1. *There exist Schwartz functions σ, ϕ on $\mathbb{R}^n$ such that*

(1) $\mathrm{supp}\,(\sigma) \subset \{\,|x| \leq 1\,\}$, $\hat{\sigma}(0) = 0$,

(2) $\mathrm{supp}\,(\hat{\phi}) \subset \{\,1/2 \leq |\xi| \leq 2\,\}$ *and* $|\hat{\phi}(\xi)| \geq c > 0$ *if* $3/5 \leq |\xi| \leq 5/3$

(3) $\sum_{\nu \in \mathbb{Z}} \hat{\sigma}\left(2^{-\nu}\xi\right) \hat{\phi}\left(2^{-\nu}\xi\right) = 1, \quad \xi \neq 0.$

For such a pair and for each $x \in \mathbb{R}^n$, *whenever* f *is a tempered distribution,* $\phi^{**} f(x) \leq C\, \sigma^{**} f(x)$, *provided* $\beta < 1$, $\lambda > 0$ *and* $0 < q \leq 1$.

The proof of this lemma is based in turn on the following

LEMMA 2.2. (cf. [4]) *Let* ϕ *be a Schwartz function on* $\mathbb{R}^n$ *such that* $\hat{\phi}$ *has a compact support away from the origin and* $\lambda > 0$. *For all integers* ν, μ,

(2-6)

$$\int |\phi_\mu * \phi_\nu(y)|\, (1 + 2^\nu |y|)^\lambda\, dy \leq C_k\, 2^{\nu-\mu} \left(1 + 2^{\nu-\mu}\right)^{-k+2m},$$

where m *is a positive integer with* $m > \lambda/2 + n/4$ *and* k *is any positive integer.*

Proof. It will be a minor modification of Lemma 4.2 in Calderón and Torchinsky [4], pp. 22–23. We claim that for any $\lambda > 0$,

$$\begin{aligned} J_\lambda &= \int |\phi_\mu * \phi_\nu(x)|^2\, (1 + 2^\nu |x|)^\lambda\, dx \\ &\leq C_k\, 2^{\nu n + 2(\nu-\mu)} \left(1 + 2^{\nu-\mu}\right)^{-2(k-2m)} \end{aligned}$$

for every positive integer k and m with $m > \lambda/4$. Once the claim were verified, then for $\delta > n$, by the Cauchy-Schwartz inequality,

$$\begin{aligned} \int |\phi_\mu * \phi_\nu(y)|\, (1 + 2^\nu |y|)^\lambda\, dy &\leq J_{2\lambda+\delta}^{1/2} \left(\int (1 + 2^\nu |y|)^{-\delta}\, dy\right)^{1/2} \\ &\leq C\, 2^{-\nu n/2}\, J_{2\lambda+\delta}^{1/2}, \end{aligned}$$

whence the assertion (2-6) follows. For the claim, changing variables, we have

$$J_\lambda = 2^{\nu n} \int |\phi * \phi_{\mu-\nu}(x)|^2\, (1 + |x|)^\lambda\, dx.$$

Observing $(1 + |x|)^\lambda \leq 2^\lambda \left(1 + |x|^{2m}\right)^2$ and invoking Plancherel's theorem, we get

$$J_\lambda \leq C\, 2^{\nu n} \int \left|(1 + (-\Delta)^m)\, \hat{\phi}(x)\, \hat{\phi}\left(2^{\nu-\mu} x\right)\right|^2 dx,$$

where Δ denotes the Laplacian. For any multi-index γ and positive integer k, note that

$$\left|D^{\gamma}\,\hat{\phi}(x)\right| \leq C_k\,(1+|x|)^{-k}, \quad \left|\hat{\phi}(x)\right| \leq C_k\,|x|\,(1+|x|)^{-k},$$

where the second inequality follows after applying the mean value theorem. Hence

$$\begin{aligned}&\left|\left(1+(-\Delta)^m\right)\,\hat{\phi}(x)\,\hat{\phi}\left(2^{\nu-\mu}\,x\right)\right|\\ &\qquad\leq C_k\,\left|2^{\nu-\mu}\,x\right|\,\left(1+2^{\nu-\mu}\,|x|\right)^{-k}\\ &\qquad\quad+\sum_{|\gamma|\leq 2m} C_{\gamma,k}\,2^{(\nu-\mu)\,|\gamma|}\,\left(1+2^{\nu-\mu}\,|x|\right)^{-k}.\end{aligned}$$

Since $\hat{\phi}$ has a compact support away from the origin, we obtain the claim upon collecting these estimates. □

Proof of Lemma 2.1. The existence of such a pair is well known. For the last estimate, start with the identity $\phi_\mu * f = \sum_{\nu\in\mathbb{Z}} (\sigma_\nu * f) * (\phi_\mu * \phi_\nu)$ to majorize

$$\begin{aligned}&2^{\mu\,\beta}\,|\,\phi_\mu * f\,(x-z)\,|\\ &\quad\leq \sum_{\nu\in\mathbb{Z}} 2^{\mu\,\beta}\int|\,\sigma_\nu * f\,(x-z-y)\,|\,|\,\phi_\mu*\phi_\nu(y)\,|\,dy\\ &\quad\leq (1+2^{\mu}\,|z|)^{\lambda}\sum_{\nu\in\mathbb{Z}}\sigma_\nu^{**}\,f\,(x) \quad\times\\ &\quad\left(1+2^{\nu-\mu}\right)^{\lambda}\,2^{-(\nu-\mu)\,\beta}\int|\,\phi_\mu*\phi_\nu\,(y)\,|\,(1+2^{\nu}\,|y|)^{\lambda}\,dy,\end{aligned}$$

where we have used the inequality (see [12])

$$\max\,(1+a+b,\,1+ab) \leq (1+a)(1+b), \quad a,\,b\geq 0.$$

It follows that $\phi_\mu^{**} f(x)$ is bounded by, according to Lemma 2.2,

$$\begin{aligned}\sum_{\nu\in\mathbb{Z}} \sigma_\nu^{**} f(x)\, \left(1+2^{\nu-\mu}\right)^\lambda 2^{-(\nu-\mu)\beta} \int |\phi_\mu * \phi_\nu(y)|\, (1+2^\nu |y|)^\lambda\, dy \\ \le C_k \sum_{\nu\in\mathbb{Z}} \sigma_\nu^{**} f(x)\, 2^{(\nu-\mu)(1-\beta)} \left(1+2^{\nu-\mu}\right)^{-k+2m+\lambda} \\ = C_k \sum_{\nu\in\mathbb{Z}} \sigma_\nu^{**} f(x)\, \Omega_{\nu-\mu}\,.\end{aligned}$$

Choosing k so large that $k > 2m+\lambda-\beta+1$, we have

$$\sum_{\mu\in\mathbb{Z}} \Omega_{\nu-\mu}^q \le \sum_{\mu\ge\nu} 2^{-(\mu-\nu)(1-\beta)q} + \sum_{\mu<\nu} 2^{(\mu-\nu)(k-2m-\lambda+\beta-1)q} \le C < \infty$$

for any $\beta < 1$ and $q > 0$. Now for $0 < q \le 1$ and for $\beta < 1$,

$$\begin{aligned}\sum_{\mu\in\mathbb{Z}} \left[\phi_\mu^{**} f(x)\right]^q &\le C \sum_{\mu\in\mathbb{Z}} \left(\sum_{\nu\in\mathbb{Z}} \sigma_\nu^{**} f(x)\, \Omega_{\nu-\mu}\right)^q \\ &\le C \sum_{\nu\in\mathbb{Z}} \left[\sigma_\nu^{**} f(x)\right]^q \left(\sum_{\mu\in\mathbb{Z}} \Omega_{\nu-\mu}^q\right) \\ &\le C \sum_{\nu\in\mathbb{Z}} \left[\sigma_\nu^{**} f(x)\right]^q,\end{aligned}$$

which yields the asserted inequality. □

LEMMA 2.3. *For the same* σ *as in* Lemma 2.1 *and for* $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, *if* $f_\alpha^\sharp \in L^p$ *for some* $p > 0$, *then for each* $x \in \mathbb{R}^n$ *and* $\lambda > n+\alpha$, $\nu \in \mathbb{Z}$,

(2-7)
$$\sigma_\nu^{**} f(x) \le C_{n,\alpha} \min\left(2^{-\nu(\alpha-\beta)} f_\alpha^\sharp(x),\, 2^{-\nu(\alpha-\beta-n/p)} \|f_\alpha^\sharp\|_p\right).$$

Proof. Making use of the inequality ([4], pp. 12)

$$(1+s)^{-\lambda} \le \sum_{k=1}^\infty 2^{-k} \chi\left(\frac{s}{2^{k/\lambda}}\right), \quad s \ge 0,\ \lambda > 0,$$

where χ denotes the characteristic function of the unit interval $[0,1]$, we obtain

$$\begin{aligned}
\sigma_\nu^{**} f(x) &= \sup_{y\in\mathbb{R}^n} 2^{\nu\beta} \, |\sigma_\nu * f(y)| \, (1+2^\nu |x-y|)^{-\lambda} \\
&\le \sum_{k=1}^{\infty} 2^{\nu\beta - k} \sup_{y\in\mathbb{R}^n} |\sigma_\nu * f(y)| \, \chi\left[\frac{|x-y|}{2^{-\nu+k/\lambda}}\right] . \qquad (2\text{-}8)
\end{aligned}$$

For each $y \in B\left(x,\, 2^{-\nu+k/\lambda}\right)$, using the fact that σ is supported in the unit ball, combined with $\hat{\sigma}(0) = 0$, we proceed to estimate

$$\begin{aligned}
&|\sigma_\nu * f(y)| \\
&\le \int_{B(y,2^{-\nu})} \left|\sigma_\nu(y-z)\, \left[f(z) - m_{2^{-\nu}(1+2^{k/\lambda})}(x)\right]\right| \, dz \\
&\le \int_{B\left(x,2^{-\nu}(1+2^{k/\lambda})\right)} \left|\sigma_\nu(y-z)\, \left[f(z) - m_{2^{-\nu}(1+2^{k/\lambda})}(x)\right]\right| \, dz \\
&\le \|\sigma\|_\infty \, 2^{-\nu\alpha} \left(1+2^{k/\lambda}\right)^{n+\alpha} f_\alpha^\sharp(x) .
\end{aligned}$$

Since the sum

$$\sum_{k=1}^{\infty} 2^{-k} \left(1+2^{k/\lambda}\right)^{n+\alpha} \le C_{n,\alpha} < \infty$$

whenever $\lambda > n+\alpha$, we get $\sigma_\nu^{**} f(x) \le C_{n,\alpha}\, 2^{-\nu(\alpha-\beta)} f_\alpha^\sharp(x)$.

To prove the other half of (2-7), for any $y \in \mathbb{R}^n$, fix a point $z \in B(y,\, 2^{-\nu})$ momentarily to get $|\sigma_\nu * f(y)| \le \|\sigma\|_\infty \, 2^{n+\alpha} \, 2^{-\nu\alpha} f_\alpha^\sharp(z)$ and then integrate over the ball $B(y,\, 2^{-\nu})$ with respect to dz after raising this inequality to the pth power to obtain $|\sigma_\nu * f(y)| \le C_{n,\alpha}\, 2^{-\nu\alpha+\nu n/p} \|f_\alpha^\sharp\|_p$. Inserting into (2-8), we are led to the desired estimate

$$\sigma_\nu^{**} f(x) \le C_{n,\alpha}\, 2^{-\nu(\alpha-\beta-n/p)} \|f_\alpha^\sharp\|_p . \qquad \square$$

Proof of Theorem A. The inequality (2-4) can be obtained from (2-5) by integration when $0 < q \leq 1$. In the case when $1 < q \leq \infty$, (2-4) follows from the case $q = 1$ by the trivial embedding

$$\|f\|_{\dot{F}_p^{\beta,q_2}} \leq \|f\|_{\dot{F}_p^{\beta,q_1}}, \quad 0 < q_1 \leq q_2 \leq \infty,\ p > 0,\ \beta \in \mathbb{R}.$$

Thus it suffices to prove the pointwise estimate (2-5). Applying Lemma 2.1 and Lemma 2.3, we observe that for any $s > 0$, $[\phi^{**} f(x)]^q$ is dominated by

$$\begin{aligned} [\sigma^{**} f(x)]^q &\leq C\,[f_\alpha^\sharp(x)]^q \sum_{\nu \geq s} 2^{-\nu(\alpha-\beta)q} + C\,\|f_\alpha^\sharp\|_p^q \sum_{\nu \leq s} 2^{-\nu(\alpha-\beta-n/p)q} \\ &\leq C\left\{ [f_\alpha^\sharp(x)]^q\, A^{q(\alpha-\beta)} + \|f_\alpha^\sharp\|_p^q\, A^{q(\alpha-\beta-n/p)} \right\}, \end{aligned}$$

where $A = 2^{-s}$ and we have used the condition $0 < \alpha - \beta < n/p$ that guarantees the convergence in the last inequality. Now set $A = \|f_\alpha^\sharp\|_p^{p/n}\,[f_\alpha^\sharp(x)]^{-p/n}$ to deduce the inequality (2-5). □

COROLLARY 2.4. *Let $0 \leq \alpha \leq 1$, $\alpha > \beta$, and $n/(n+\alpha) < p < n/(\alpha-\beta)$. Assume that $|\{|f| > \epsilon\}| < \infty$ for every $\epsilon > 0$. If $f_\alpha^\sharp \in L^p$, then*

(2-8) $$\|f\|_{\dot{B}_r^{\beta,q}} \leq C\,\|f_\alpha^\sharp\|_p, \quad 1/r = 1/p - (\alpha-\beta)/n,$$

for each $r < q \leq \infty$.

The assertion follows immediately from Theorem A in view of the embedding inequality

$$\|f\|_{\dot{B}_p^{\beta,q}} \leq \|f\|_{\dot{F}_p^{\beta,q}}, \quad 0 < p < q \leq \infty,\ \beta \in \mathbb{R},$$

which essentially results from the integral inequality of Minkowski.

COROLLARY 2.5. *Suppose that $|\{|f| > \epsilon\}| < \infty$ for every $\epsilon > 0$. Let $0 \leq \beta < n$ and $f^\sharp \in L^p$ for $1 < p < n/\beta$.*(3) *Then for all $0 < q \leq \infty$,*

(2-10) $$\|f\|_{\dot{F}_r^{-\beta,q}} \leq C\,\|f^\sharp\|_p, \quad 1/r = 1/p - \beta/n.$$

(3) If $\beta = 0$, then we take $1 < p < \infty$.

REMARK 2. Taking $\beta = 0$ and $q = 2$, we get (1-4), the inequality of C. Fefferman and E. M. Stein. This inequality is also proved by Calderón and Scott [3]. Proofs of the two are based on weak-type estimates involving Hardy-Littlewood maximal function.

We finally state the L^p version of Theorem A to see how it improves the result (1-3) of Calderón and Scott.

COROLLARY 2.6. *Suppose that* $|\{|f| > \epsilon\}| < \infty$ *for every* $\epsilon > 0$ *and* $f_\alpha^\sharp \in L^p$ *for* $0 < \alpha \leq 1$ *and* $n/(n+\alpha) < p < n/\alpha$. *Then*

$$\|f\|_r \leq C\,\|f_\alpha^\sharp\|_p, \quad 1/r = 1/p - \alpha/n. \tag{2-11}$$

3. BMO and Lipschitz estmates

The foregoing machinery breaks down when $p \geq n/(\alpha - \beta)$. An inspection shows that the number $p = n/\alpha$ plays the role of *critical index* (*or end point*), which makes our consideration of β in the cases $p \geq n/\alpha$ unnecessary. However, we observe from (2-7) that

$$\|f\|_{\dot F_\infty^{(\alpha-n/p),\infty}} = \|f\|_{\dot\Lambda_{(\alpha-n/p)}} \leq C\,\|f_\alpha^\sharp\|_p, \quad n/\alpha < p \leq \infty. \tag{3-1}$$

Moreover, if f satisfies that for $0 < \alpha < 1$,

$$|f(x) - f(y)| \leq B\,|x - y|^\alpha \quad \text{for all} \quad x, y \in \mathbb{R}^n, \tag{3-2}$$

then it is straightforward to note that $\|f_\alpha^\sharp\|_\infty \leq C_n B$ and thus $\|f_\alpha^\sharp\|_\infty \leq C_n\,\|f\|_{\dot\Lambda_\alpha}$. Consequently, we have the following result:

THEOREM B. *Suppose that* $|\{|f| > \epsilon\}| < \infty$ *for every* $\epsilon > 0$. *Let* $0 < \alpha \leq 1$ *and* $f_\alpha^\sharp \in L^p$ *for* $n/\alpha < p \leq \infty$. *Then* $f \in \dot\Lambda_{(\alpha-n/p)}$ *with*

$$\|f\|_{\dot\Lambda_{(\alpha-n/p)}} \leq C\,\|f_\alpha^\sharp\|_p. \tag{3-3}$$

Furthermore, in the case of $0 < \alpha < 1$, *we have the equivalence*

$$\|f_\alpha^\sharp\|_\infty \sim \|f\|_{\dot\Lambda_\alpha}. \tag{3-4}$$

It remains to examine the critical case $p = n/\alpha$.

THEOREM C. *Suppose that* $\big|\{\,|f| > \epsilon\,\}\big| < \infty$ *for every* $\epsilon > 0$. *Let* $0 < \alpha \le 1$ *and assume that* $f^\sharp_\alpha \in L^p(\mathbb{R}^n)$ *for* $p = n/\alpha$. *Then* $f \in BMO(\mathbb{R}^n)$ *with*

$$\|f\|_{\mathrm{BMO}} \le C_{n,\alpha}\, \|f^\sharp_\alpha\|_p\,. \tag{3-5}$$

Proof. Let $x \in \mathbb{R}^n$ and $\rho > 0$. Fix a point $z \in B(x,\rho)$ momentarily. It follows from the inclusion $B(x,\rho) \subset B(z,2\rho)$ that

$$\begin{aligned}\int_{B(x,\rho)} \big|\, f(y) - m_\rho(x)\,\big|\, dy &\le (1+2^n) \int_{B(z,2\rho)} \big|\, f(y) - m_{2\rho}(z)\,\big|\, dy \\ &\le 2^{\alpha+n}\,(1+2^n)\,\rho^{\alpha+n}\, f^\sharp_\alpha(z)\,.\end{aligned}$$

As before, raise this inequality to the power p and integrate over $B(x,\rho)$ with respect to dz to obtain

$$\rho^{-n} \int_{B(x,\rho)} \big|\, f(y) - m_\rho(x)\,\big|\, dy \le C_{n,\alpha}\, \|f^\sharp_\alpha\|_p\,,$$

whence $f^\sharp(x) \le C_{n,\alpha}\, \|f^\sharp_\alpha\|_p$ and the desired result follows. □

We end this paper with listing a couple of applications. On account of our discussions in the introduction, we first have

THEOREM D. *Let* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ *with* $\nabla f \in H^p(\mathbb{R}^n)$ *for* $n/(n+1) < p < \infty$.

(D1) *If* $n/(n+1) < p < n/(1-\beta)$ *for* $-n < \beta < 1$ *and* $1/r = 1/p - (1-\beta)/n$, *then*

$$\|f\|_{\dot F^{\beta,q}_r} \le C\, \|\,\nabla f\,\|_{H^p} \quad \text{for all} \quad 0 < q \le \infty\,,$$
$$\|f\|_{\dot B^{\beta,q}_r} \le C\, \|\,\nabla f\,\|_{H^p} \quad \text{for all} \quad r < q \le \infty\,.$$

(D2) *If* $p = n$, *then* $\|f\|_{\mathrm{BMO}} \le C\, \|\,\nabla f\,\|_{H^n}$.

(D3) *If* $n < p < \infty$, *then* $\|f\|_{\dot\Lambda_{(1-n/p)}} \le C\, \|\,\nabla f\,\|_{H^p}$.

Next with the notation

$$|f|_{\alpha,p} = \left(\iint_{\mathbb{R}^n\times\mathbb{R}^n} \frac{|\,f(x) - f(y)\,|^p}{|x-y|^{n+\alpha p}}\, dx\, dy\right)^{1/p}, \quad 1 \le p < \infty \tag{3-6}$$

and the obvious modification when $p = \infty$, we have

THEOREM E. *Let* $0 < \alpha < 1$ *and assume* $f \in \dot{W}^{\alpha,p}(\mathbb{R}^n)$ *for* $1 \leq p \leq \infty$.

(E1) *If* $1 \leq p < n/(\alpha - \beta)$ *for* $-n + \alpha < \beta < \alpha$ *and* $1/r = 1/p - (\alpha - \beta)/n$, *then*

$$\|f\|_{\dot{F}_r^{\beta,q}} \leq C\left|f\right|_{\alpha,p} \quad \text{for all} \quad 0 < q \leq \infty,$$
$$\|f\|_{\dot{B}_r^{\beta,q}} \leq C\left|f\right|_{\alpha,p} \quad \text{for all} \quad r < q \leq \infty.$$

(E2) *If* $p = n/\alpha$, *then* $\|f\|_{\mathrm{BMO}} \leq C\left|f\right|_{\alpha,n/\alpha}$.

(E3) *If* $n/\alpha < p \leq \infty$, *then* $\|f\|_{\dot{\Lambda}_{(\alpha-n/p)}} \leq C\left|f\right|_{\alpha,p}$.

EPILOGUE. To our great sorrow, Professor Ham passed away on February 2, 1998 at the age of 60. With many collegues and students, we deeply honor what he had done for us, especially his teaching and friendship, which have been influencing us and will do so forever.

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