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ON DERIVATIONS IN NONCOMMUTATIVE SEMISIMPLE BANACH ALGEBRAS

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ABSTRACT. The purpose of this paper is to prove the following results: Let A be a noncommutative semisimple Banach algebra. (1) Suppose that a linear derivation $D : A \to A$ is such that $[D(x), x]x = 0$ holds for all $x \in A$. Then we have $D = 0$. (2) Suppose that a linear derivation $D : A \to A$ is such that $D(x)x^2 + x^2D(x) = 0$ holds for all $x \in A$. Then we have $D = 0$.

1. Introduction

Throughout this paper R will represent an associative ring with center $Z(R)$, and A will represent an algebra over a complex field. The commutator $xy - yx$ will be denoted by $[x, y]$. We make use of the basic commutator identities $[xy, z] = [x, z]y + x[y, z]$, $[x, yz] = [x, y]z + y[x, z]$. An additive mapping D from R to R is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all $x, y \in R$. A derivation D is inner if there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. Recall that a ring R is prime if $aRb = (0)$ implies that either $a = 0$ or $b = 0$. Sinclair [1] has proved that every linear derivation on a semisimple Banach algebra is continuous. Singer and Wermer [4] state that every continuous linear derivation on a commutative Banach algebra maps the algebra into its Jacobson radical. Combining these two results we obtain that there are no nonzero derivations on a commutative semisimple Banach algebra. Now it seems natural to ask, under what additional assumptions a linear derivation on a noncommutative semisimple Banach algebra is zero. It is our aim in this paper to give answers to the question above.

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2. The results

We now state and prove the main results.

THEOREM 2.1. *Let A be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $D : A \to A$ such that $[D(x), x]x = 0$ holds for all $x \in A$. Then we have $D = 0$.*

Proof of Theorem 2.1. For the proof of the theorem we shall need the following purely algebraic result which can be proved without any specific knowledge concerning prime rings. □

LEMMA 2.1. *Let R be a noncommutative prime ring of characteristic different from two and I a nonzero two-sided ideal of R. Suppose that there exists a derivation $D : R \to R$ such that $[D(x), x]x = 0$ holds for all $x \in I$. Then we have $D = 0$ on R.*

Proof of Lemma 2.1. We define a mapping $B(.,.) : I \times I \to I$ by the relation

$$B(x,y) = [D(x), y] + [D(y), x],\ x, y \in I. \tag{1}$$

Obviously, $B(x, y) = B(y, x)$ for all $x, y \in I$ and $B(.,.)$ is additive in both arguments. Moreover, a simple calculation shows that the relation

$$B(xy, z) = B(x, z)y + xB(y, z) + D(x)[y, z] + [x, z]D(y) \tag{2}$$

holds for all $x, y, z \in I$. We shall write $f(x)$ for $B(x, x)$. Then

$$f(x) = 2[D(x), x],\ x \in I. \tag{3}$$

It is easy to see that

$$f(x + y) = f(x) + f(y) + 2B(x, y) \tag{4}$$

is fulfilled for all $x, y \in I$. Now the assumption of the lemma can be written as follows

$$f(x)x = 0,\ x \in I. \tag{5}$$

The linearization of (5) gives

$$\begin{aligned} 0 &= (f(x)+f(y)+2B(x,y))(x+y) \\ &= f(x)x+f(y)x+2B(x,y)x+f(x)y \\ &\quad + f(y)y+2B(x,y)y, \end{aligned} \tag{6}$$

which reduces to

$$f(x)y+f(y)x+2B(x,y)x+2B(x,y)y=0,\ x,y\in I. \tag{7}$$

Replacing x by $-x$ in (7), and subtracting the new result from (7), we have

$$f(x)y+2B(x,y)x=0,\ x,y\in I, \tag{8}$$

since R is of characteristic not two. Let y be yx in (8). Then, by (5) and (8), we get

$$\begin{aligned} 0 &= f(x)yx+2B(x,yx)x \\ &= 2[y,x]D(x)x,\ x,y\in I. \end{aligned} \tag{9}$$

Hence we arrive at

$$[y,x]D(x)x=0,\ x,y\in I, \tag{10}$$

since R is of characteristic not two. We intend to prove that

$$D(x)x=0 \tag{11}$$

holds for all $x\in I$. Suppose on the contrary that $D(a)a\neq 0$ for some $a\in I$. Note that I is a non-zero noncommutative prime ring of characteristic not two and a mapping $y\mapsto [y,a]$ is an inner derivation on I. Then (10)and Lemma 1 in [2] imply $a\in Z(I)$. We have therefore proved that $D(x)x=0$ in case $x\notin Z(I)$. It remains to prove that $D(x)x=0$ also in the case when $x\in Z(I)$. Let $x\in Z(I)$ and let $y\notin Z(I)$. We have also $x+y\notin Z(I)$. We see that $D(y)y=0$ and

$D(x+y)(x+y) = 0$. Then $0 = (D(x)+D(y))(x+y) = D(x)x + D(x)y + D(y)x + D(y)y = D(x)x + D(x)y + D(y)x$. Hence

$$D(x)x + D(x)y + D(y)x = 0. \tag{12}$$

Replacing x by $-x$ in (12), we have

$$D(x)x - D(x)y - D(y)x = 0. \tag{13}$$

From (12) and (13) it follows $D(x)x = 0$, which completes the proof of (11). The linearization of (11) leads to

$$D(x)y + D(y)x = 0, \ x, y \in I. \tag{14}$$

Substituting zy for y in (14), we get

$$\begin{aligned} 0 &= D(x)zy + D(zy)x \\ &= D(x)zy + zD(y)x + D(z)yx, \ x, y \in I, \ z \in R. \end{aligned} \tag{15}$$

Combining (14) with (15), we obtain

$$[D(x), z]y + D(z)yx = 0, \ x, y \in I, \ z \in R. \tag{16}$$

Replacing z by $D(x)$ in (16), we have

$$D^2(x)yx = 0, \ x, y \in I.$$

Since I is prime, we know that $D^2(x) = 0$ holds for all $x \in I$. This yields $D(x) = 0$ for all $x \in I$ by Theorem 1 in [2]. Now, substituting rx ($r \in R$) for x, we have $D(r)x = 0$, that is, $D(r)I = 0$. Since R is prime and I is nonzero, it follows that $D(r) = 0$ for all $r \in R$. The proof of Lemma 2.1 is complete. $\square$

Proof of Theorem 2.1 *continued.* By the result of Johnson and Sinclair [1] every linear derivation on a semisimple Banach algebra is continuous. Sinclair [3] has proved that every continuous linear derivation on a Banach algebra leaves the primitive ideals of A invariant. Hence for any primitive ideal $P \subset A$,we can define a linear derivation $D_P : A/P \to A/P$ by $D_P(x+P) = D(x) + P,\ x \in A$. The assumption of the theorem $[D(x), x]x = 0, x \in A$ gives $[D_P(x+P), x+P](x+P) = P, x \in A$. Since P is a primitive ideal, A/P is prime. Hence, in case A/P is noncommutative, we have $D_P = 0$, since all the assumptions of Lemma 2.1 are fulfilled. In case A/P is commutative, we can conclude that $D_P = 0$ as well since A/P is semisimple and since we know that there are no nonzero linear derivations on commutative semisimple Banach algebras. This implies that $D(x)$ is in the intersection of all primitive ideals of A for all $x \in A$. Since the intersection of all primitive ideals is the Jacobson radical, and A is semisimple, it follows $D = 0$. The proof of Theorem 2.1 is complete.

As a special case of Theorem 2.1 we obtain the following result which characterizes commutative semisimple Banach algebras among all semisimple Banach algebras. □

COROLLARY 2.1. *Let A be a semisimple Banach algebra. Suppose that $[[x,y],x]x = 0$ holds for all $x, y \in A$. Then A is commutative.*

THEOREM 2.2. *Let A be a noncommutative semisimple Banach algebra. Suppose that there exists a linear derivation $D : A \to A$ such that $D(x)x^2 + x^2D(x) = 0$ holds for all $x \in A$. Then we have $D = 0$.*

Proof of Theorem 2.2. For the proof of Theorem 2.2 as in Theorem 2.1 we also need prove the following algebraic result. □

LEMMA 2.2. *Let R be a noncommutative prime ring of characteristic different from two and I a nonzero two-sided ideal of R. Suppose that there exists a derivation $D : R \to R$ such that $D(x)x^2 + x^2D(x) = 0$ holds for all $x \in I$. Then we have $D = 0$ on R.*

Proof of Lemma 2.2. Suppose that

$$D(x)x^2 + x^2D(x) = 0 \tag{1}$$

holds for all $x \in I$. The linearization of (1) leads to

$$\begin{aligned} 0 = {} & D(x)xy + D(x)yx + D(x)y^2 + D(y)x^2 + D(y)xy \\ & + D(y)yx + x^2D(y) + xyD(x) + xyD(y) \\ & + yxD(x) + yxD(y) + y^2D(x), \ x, y \in I. \end{aligned} \tag{2}$$

Replacing y by $-y$ in (2), and subtracting the result from (2), we have

$$\begin{aligned} & D(x)xy + D(x)yx + D(y)x^2 + x^2D(y) \\ & + xyD(x) + yxD(x) = 0, \ x, y \in I, \end{aligned} \tag{3}$$

since R is of characteristic not two. Substituting xy for y in (3) and using (1), we arrive at

$$\begin{aligned} & D(x)xyx + xD(x)x^2 + D(x)yx + x^3D(y) \\ & + x^2yD(x) + xyxD(x) = 0, \ x, y \in I. \end{aligned} \tag{4}$$

Left multiplication of (3) by x leads to

$$\begin{aligned} & xD(x)xy + xD(x)yx + xD(y)x^2 + x^3D(y) \\ & + x^2yD(x) + xyxD(x) = 0, \ x, y \in I. \end{aligned} \tag{5}$$

Subtracting (5) from (4), we obtain

$$-xD(x)xy + [D(x), x]yx + D(x)yx^2 = 0, \ x, y \in I. \tag{6}$$

Replacing y by $yD(x)$ in (6), we have

$$-xD(x)xyD(x) + [D(x), x]yD(x)x + D(x)yD(x)x^2 = 0, \ x, y \in I. \tag{7}$$

Right multiplication of (6) by $D(x)$ gives

$$-xD(x)xyD(x) + [D(x), x]yxD(x) + D(x)yx^2D(x) = 0, \ x, y \in I. \tag{8}$$

Subtracting (8) from (7), we obtain

$$D(x)y[D(x), x^2] + [D(x), x]y[D(x), x] = 0, \ x, y \in I. \tag{9}$$

Putting xy instead of y in (9), it follows that

$$D(x)xy[D(x),x^2]+[D(x),x]xy[D(x),x]=0,\ x,y\in I. \tag{10}$$

Left multiplication of (10) by x gives

$$xD(x)y[D(x),x^2]+x[D(x),x]y[D(x),x]=0,\ x,y\in I. \tag{11}$$

Subtracting (11) from (10), we obtain

$$\begin{aligned}&[D(x),x]y[D(x),x^2]\\&\quad+[[D(x),x],x]y[D(x),x]=0,\ x,y\in I.\end{aligned} \tag{12}$$

Replacing y by $y[D(x),x]z$ in (12), we get

$$\begin{aligned}&[D(x),x]y[D(x),x]z[D(x),x^2]\\&\quad+[[D(x),x],x]y[D(x),x]z[D(x),x]=0,\ x,y,z\in I.\end{aligned}$$

Using (12) we can write $-[[D(x),x]x]z[D(x),x]$ for $[D(x),x]z[D(x),x^2]$ and $-[D(x),x]y[D(x),x^2]$ for $[[D(x),x],x]y[D(x),x]$ in the above calculation.

Hence we arrive at

$$\begin{aligned}&[D(x),x]y[[D(x),x],x]z[D(x),x]\\&\quad+[D(x),x]y[D(x),x^2]z[D(x),x]=0,\end{aligned}$$

which can be reduced in the form

$$[D(x),x]y[D(x),x]xz[D(x),x]=0,\ x,y,z\in I.$$

Since I is a prime ring, it follows that

$$[D(x),x]x=0$$

holds for all $x\in I$. This implies that $D=0$ on R by Lemma 2.1. The proof of Lemma 2.2 is complete. □

Proof of Theorem 2.2 *continued.* Let us use the same argument as that used to prove Theorem 2.1. Then we can define a linear derivation $D_P : A/P \to A/P$, where P is any primitive ideal of A, by $D_P(x+P) = D(x) + P$, $x \in A$. Also the assumption of Theorem 2.2 $D(x)x^2 + x^2D(x) = 0$, $x \in A$ gives $D_P(x+P)(x+P)^2 + (x+P)^2D_P(x+P) = P$, $x \in A$. Hence Lemma 2.2 deduces $D_P = 0$, and semisimplicity of A forces $D = 0$. The proof of Theorem 2.2 is complete. □

We also obtain the following result as a special case of Theorem 2.2 as in Corollary 2.1.

COROLLARY 2.2. *Let A be a semisimple Banach algebra. Suppose that $[x,y]x^2+x^2[x,y] = 0$ holds for all $x, y \in A$. Then A is commutative.*

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