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SOME CRITERIA FOR p-VALENT FUNCTIONS

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ABSTRACT. The object of the present paper is to derive some sufficient conditions for p-valently close-to-convexity, p-valently starlikeness and p-valently convexity.

1. Introduction

Let $A(p)$ be the class of functions of the form

$$f(z) = z^p + \sum_{m=p+1}^{\infty} a_m z^m \quad (p \in N = \{1, 2, 3, \cdots\})$$

which are analytic in the unit disk $E = \{z : \mid z \mid < 1\}$.

A function $f(z)$ in $A(p)$ is said to be p-valently convex if and only if

$$Re\{1 + \frac{zf''(z)}{f'(z)}\} > 0 \quad (z \in E).$$

We denote by $C(p)$ the subclass of $A(p)$ consisting of all p-valently convex functions in E. A function $f(z)$ in $A(p)$ is said to be p-valently starlike if and only if

$$Re\frac{zf'(z)}{f(z)} > 0 \quad (z \in E).$$

We denote by $S(p)$ the subclass of $A(p)$ consisting of all p-valently starlike functions in E. A function $f(z)$ in $A(p)$ is said to be p-valently

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close-to-convex, if there exists a p-valently starlike function $g(z) \in S(p)$ for which $f(z)$ satisfies

$$Re\frac{zf'(z)}{g(z)} > 0 \quad (z \in E).$$

We denote by $K(p)$ the subclass of $A(p)$ consisting of functions which are p-valently close-to-convex in E.

It is well known that every p-valently close-to-convex function is p-valent in E and that $C(p) \subset S(p) \subset K(p)$.

2. Preliminaries

In order to derive our results, we need the following lemmas.

LEMMA 1 ([2]). *Let $h(z)$ be analytic and convex univalent in E, $h(0) = 1$, and let $g(z) = 1 + b_n z^n + b_{n+1} z^{n+1} + \cdots$ $(n \in N)$ be analytic in E. if*

$$g(z) + \frac{1}{c} zg'(z) \prec h(z),$$

then

$$g(z) \prec \frac{c}{n} z^{-\frac{c}{n}} \int_0^z t^{\frac{c}{n}-1} h(t) dt,$$

where the symbol $\prec$ stands for subordination, $c \neq 0$ and Re $c \geqslant 0$.

LEMMA 2 ([3]). *Let $w(z) = w_n z^n + w_{n+1} z^{n+1} + \cdots$ be analytic in $|z| < r$ with $w(z) \not\equiv 0$. If there exists a point z_1, $0 < |z_1| < r$, such that*

$$\max_{|z| \leqslant |z_1|} |w(z)| = |w(z_1)|,$$

then $z_1 w'(z_1) = \lambda w(z_1)$, where $\lambda \geqslant n \geqslant 1$.

LEMMA 3 ([1]). *Suppose that $f(z)$ and $g(z)$ are analytic in E, $f(0) = g(0) = 0$, and $g(z)$ maps E onto a (possibly many-sheeted) region which is starlike with respect to the origin. If Re $\{f'(z)/g'(z)\} > 0$ in E, then*

$$Re\{f(z)/g(z)\} > 0 \quad (z \in E).$$

LEMMA 4. *If $\alpha > 0$ and $f(z) \in A(p)$ satisfies*

$$\left|(1-\alpha)\frac{zf'(z)}{f(z)} + \alpha\left(1 + \frac{zf''(z)}{f'(z)}\right) - p\right| < p + \alpha \quad (z \in E),$$

then $f(z) \in S(p)$.

The above lemma is due to Yang [10, Corollary 2].

LEMMA 5. *If $f(z) \in A(p)$ satisfies*

$$Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} < p + \frac{1}{2} \quad (z \in E),$$

then $f(z) \in S(p)$.

We owe this lemma to Owa [8].

3. Main results

THEOREM 1. *Let $f(z) = z^p + \sum_{m=p+n}^{\infty} a_m z^m$ $(n \in N)$ be analytic in E and suppose that there exists a positive integer k for which*

$$\left(\frac{f^{(k)}(z)}{z^{p-k-1}}\right)' \prec \frac{p!}{(p-k)!}\left(\frac{1+Az}{1+Bz}\right), \tag{1}$$

where $1 \leqslant k \leqslant p$, $-1 \leqslant B < 1$ and

$$A = A(n, B) = \frac{\int_0^1 \frac{u^{1/n-1}}{1-Bu}du}{\int_0^1 \frac{u^{1/n}}{1-Bu}du}. \tag{2}$$

Then $f(z) \in K(p)$.

Proof. It is clear that $A - B > 0$ for A given by (2). Let us put

$$g(z) = \frac{(p-k)!}{p!} \; \frac{f^{(k)}(z)}{z^{p-k}}. \tag{3}$$

Then $g(z) = 1 + b_n z^n + \cdots$ is analytic in E and it follows from (1) that

$$g(z) + zg'(z) \prec \frac{1+Az}{1+Bz}.$$

Since $h(z) = (1+Az)/(1+Bz)$ is convex univalent in E, an application of Lemma 1 yields

$$g(z) \prec \frac{1}{n} \int_0^1 u^{\frac{1}{n}-1} \left(\frac{1+Auz}{1+Buz} \right) du. \tag{4}$$

In view of $-1 \leq B < 1$ and $A > B$, from(3), (4) and (2) we obtain

$$Re \frac{f^{(k)}(z)}{z^{p-k}} > \frac{p!}{(p-k)!} \; \frac{1}{n} \int_0^1 u^{1/n-1} \left(\frac{1-Au}{1-Bu} \right) du = 0 \quad (z \in E). \tag{5}$$

Applying Lemma 3 repeatedly, it follows from (5) that

$$Re \frac{zf'(z)}{g(z)} > 0 \quad (z \in E),$$

where $g(z) = z^p \in S(p)$. Hence $f(z) \in K(p)$ and the proof is complete. □

Putting $n = 1, \; B = -1$ and $k = p - 1 \geq 1$ in Theorem 1, we have

COROLLARY 1. *Let $p \geqslant 2$. If $f(z) \in A(p)$ satisfies*

$$Ref^{(p)}(z) > -p! \frac{2\log 2 - 1}{2(1-\log 2)} \quad (z \in E),$$

then $f(z) \in K(p)$.

Taking $B = 0$ and $k = p - 1 \geqslant 1$, Theorem 1 reduces to

COROLLARY 2. *Let* $p \geqslant 2$. *If* $f(z) = z^p + \sum_{m=p+n}^{\infty} a_m z^m$ $(n \in N)$ *is analytic in E and satisfies*

$$| f^{(p)}(z) - p! | < (n+1)(p!) \quad (z \in E),$$

then $f(z) \in K(p)$.

REMARK 1. For $n = 1$, Corollary 2 was also proved by Nunokawa et al [7, Theorem 2].

COROLLARY 3. *Let* $p \geqslant 2$. *If* $f(z) = z^p + \sum_{m=p+n}^{\infty} a_m z^m$ $(n \in N)$ *is analytic in E and satisfies*

$$\text{(6)} \quad | f^{(p)}(z) + czf^{(p+1)}(z) - p! | < (n+1) \mid nc + 1 \mid (p!) \quad (z \in E),$$

where $c \neq 0$ *and* $Re\, c \geqslant 0$, *then* $f(z) \in K(p)$.

Proof. Putting $g(z) = f^{(p)}(z)/p!$, *then* $g(z) = 1 + b_n z^n + \cdots$ is analytic in E and it follows from (6) that

$$g(z) + czg'(z) \prec 1 + (n+1)(nc+1)z.$$

Hence, by using Lemma 1, we have $g(z) \prec 1 + (n+1)z$, i.e.

$$| f^{(p)}(z) - p! | < (n+1)(p!) \quad (z \in E).$$

Thus by Corollary 2, we conclude that $f(z) \in K(p)$. □

REMARK 2. When $n = 1$ and $c = Rec \to +\infty$ in Corollary 3, we get Theorem 3 of [7].

A function $f(z) \in A(p)$ is said to be in the class $S(p, \alpha)(0 \leqslant \alpha < 1)$ if it satisfies

$$Re \frac{zf'(z)}{f(z)} > p\alpha \quad (z \in E).$$

Clearly $S(p, \alpha) \subset S(p)$ for $0 \leqslant \alpha < 1$.

In [9] the following lemma was proved.

LEMMA A. *Let $\beta \geqslant 0,\ 0 \leqslant \alpha \leqslant 1/2,$ and let $f(z) \in A(1)$ satisfy*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < (1-\alpha)^{(1-2\beta)} \left(1 - \frac{3}{2}\alpha + \alpha^2\right)^{\beta}$$

for $z \in E$. Then $f(z) \in S(1,\alpha)$.

We now derive

THEOREM 2. *Let $\beta \geqslant 0,\ 0 \leqslant \alpha < 1,$ and let $f(z) = z^p + \sum_{m=p+n}^{\infty} a_m z^m$ $(n \in N)$ be analytic in E and satisfy*

$$\left|\frac{zf'(z)}{f(z)} - p\right|^{1-\beta} \left|1 + \frac{zf''(z)}{f'(z)} - p\right|^{\beta}$$

$$< \begin{cases} p^{1-\beta} \quad (1-\alpha)(p + \frac{n}{2(1-\alpha)})^{\beta} \quad (0 \leqslant \alpha \leqslant \frac{1}{2}), & (7)\\ p^{1-\beta} \quad (1-\alpha)(p+n)^{\beta} \quad (\frac{1}{2} < \alpha < 1) & (8)\end{cases}$$

for $z \in E$. Then $f(z) \in S(p,\alpha)$.

Proof. We first consider the case $0 \leqslant \alpha \leqslant 1/2$. Define a function $w(z)$ by

$$\frac{zf'(z)}{f(z)} = p\alpha + p(1-\alpha)\frac{1+w(z)}{1-w(z)}. \tag{9}$$

Then $w(z)$ is either analytic or meromorphic in E and $w^{(k)}(0) = 0$ $(0 \leqslant k \leqslant n-1)$.

From (9) we can get

$$1 + \frac{zf''(z)}{f'(z)} - p = \frac{2(1-\alpha)}{1-w(z)}\left\{pw(z) + \frac{zw'(z)}{1+(1-2\alpha)w(z)}\right\},$$

whence

$$\begin{aligned} &\left|\frac{zf'(z)}{f(z)} - p\right|^{1-\beta} \left|1 + \frac{zf''(z)}{f'(z)} - p\right|^{\beta} \\ &= 2p^{1-\beta}(1-\alpha)\left|\frac{w(z)}{1-w(z)}\right| \left|p + \frac{zw'(z)}{w(z)}\left(\frac{1}{1+(1-2\alpha)w(z)}\right)\right|^{\beta}. \end{aligned} \tag{10}$$

We claim that $|w(z)| < 1$ for $z \in E$. Otherwise there exists a point $z_1 \in E$ such that $\max_{|z| \leq |z_1|} |w(z)| = |w(z_1)| = 1$. By Lemma 2, we can write $z_1 w'(z_1) = \lambda w(z_1) (\lambda \geqslant n)$. Thus it follows from (10) that

$$\begin{aligned} &\left| \frac{z_1 f'(z_1)}{f(z_1)} - p \right|^{1-\beta} \left| 1 + \frac{z_1 f''(z_1)}{f'(z_1)} - p \right|^{\beta} \\ &\geqslant p^{1-\beta}(1-\alpha) \left(p + Re \frac{\lambda}{1 + (1-2\alpha) w(z_1)} \right)^{\beta} \\ &\geqslant p^{1-\beta}(1-\alpha) \left(p + \frac{n}{2(1-\alpha)} \right)^{\beta} \end{aligned}$$

for $0 \leqslant \alpha \leqslant 1/2$. This contradicts (7) and hence $|w(z)| < 1$ $(z \in E)$. From (9) we see that $f(z) \in S(p, \alpha)$.

Next consider the case $1/2 < \alpha < 1$. The function $w(z)$ defined by

$$\frac{zf'(z)}{f(z)} = \frac{p}{1 - (1/\alpha - 1) w(z)} \tag{11}$$

is either analytic or meromorphic in E and $w^{(k)}(0) = 0$ $(0 \leq k \leq n-1)$.

It follows from (11) that

$$\begin{aligned} &\left| \frac{zf'(z)}{f(z)} - p \right|^{1-\beta} \left| 1 + \frac{zf''(z)}{f'(z)} - p \right|^{\beta} \\ &= p^{1-\beta}(1/\alpha - 1) \left| \frac{w(z)}{1 - (1/\alpha - 1) w(z)} \right| \left| p + \frac{zw'(z)}{w(z)} \right|^{\beta}. \end{aligned} \tag{12}$$

Suppose that there exists a point $z_1 \in E$ such that $\max_{|z| \leq |z_1|} |w(z)| = |w(z_1)| = 1$. Then Lemma 2 gives $z_1 w'(z_1) = \lambda w(z_1)$ $(\lambda \geqslant n)$. Therefore, (12) leads to

$$\begin{aligned} &\left| \frac{z_1 f'(z_1)}{f(z_1)} - p \right|^{1-\beta} \left| 1 + \frac{z_1 f''(z_1)}{f'(z_1)} - p \right|^{\beta} \\ &\geqslant p^{1-\beta} \frac{1/\alpha - 1}{1 + |1/\alpha - 1|} (p + \lambda)^{\beta} \\ &\geqslant p^{1-\beta}(1-\alpha)(p+n)^{\beta}, \end{aligned}$$

which contradicts (8). Thus $| w(z) | < 1$ $(z \in E)$ and it follows from (11) that $f(z) \in S(p, \alpha)$. This completes the proof of Theorem 2. □

For $p = n = 1$, Theorem 2 reduces to

COROLLARY 4. *Let $\beta \geqslant 0$, $0 \leqslant \alpha < 1$, and let $f(z) \in A(1)$ satisfy*

$$\left|\frac{zf'(z)}{f(z)} - 1\right|^{1-\beta} \left|\frac{zf''(z)}{f'(z)}\right|^{\beta} < \begin{cases} (1-\alpha)^{1-\beta} \quad (3/2-\alpha)^{\beta} & (0 \leqslant \alpha \leqslant 1/2), \\ 2^{\beta}(1-\alpha) & (1/2 < \alpha < 1) \end{cases}$$

for $z \in E$. Then $f(z) \in S(1, \alpha)$.

REMARK 3. Since

$$(1-\alpha)^{1-\beta}(3/2-\alpha)^{\beta} > (1-\alpha)^{1-2\beta}(1-3\alpha/2+\alpha^2)^{\beta}$$

for $0 \leqslant \alpha < 1/2$ and $\beta > 0$, Corollary 4 is an improvement and extension of Lemma A.

By applying Lemma A, Owa and Srivastava [9] have obtained some results involving starlike generalized hypergeometric functions of order α for $0 \leqslant \alpha \leqslant 1/2$. Now we can extend their results with the help of Corollary 4. For example, setting $\beta = 1$ in Corollary 4, we easily have

COROLLARY 5. *Let the generalized hypergeometric function ${}_pF_q(z)$ be defined by*

$${}_pF_q(z) = \sum_{n=0}^{\infty} \frac{(a_1)_n (a_2)_n \cdots (a_p)_n}{(b_1)_n (b_2)_n \cdots (b_q)_n} \frac{z^n}{n!} \quad (p \leqslant q+1),$$

where $(a)_0 = 1, (a)_n = a(a+1) \cdots (a+n-1)$ $(n \in N)$, and $b_j \neq 0, -1, -2, \cdots$ $(1 \leqslant j \leqslant q)$. If $\Pi_{j=1}^{p} a_j \neq 0$ and

$$\left|\frac{z \cdot {}_pF_q''(z)}{{}_pF_q'(z)}\right| < \begin{cases} 3/2-\alpha & (0 \leqslant \alpha \leqslant 1/2) \\ 2(1-\alpha) & (1/2 < \alpha < 1) \end{cases}$$

for $z \in E$, then ${}_pF_q(z)$ is starlike of order α with respect to 1, that is ,

$$\left(\Pi_{j=1}^{q} b_j / \Pi_{j=1}^{p} a_j\right) \left({}_pF_q(z) - 1\right) \in S(1, \alpha).$$

REMARK 4. Corollary 5 improves and extends Theorem 3 of [9].

THEOREM 3. *Let $f(z) \in A(p)$ and suppose that there exists a posi tive integer k for which*

$$\left|(1-\alpha)\frac{zf^{(k)}(z)}{f^{(k-1)}(z)} + \alpha\left(1+\frac{zf^{(k+1)}(z)}{f^{(k)}(z)}\right) - (p-k+1)\right| < p-k+1+\alpha \quad (z \in E), \tag{13}$$

where $2 \leqslant k \leqslant p$ and $\alpha > 0$. Then we have

$$Re\frac{zf''(z)}{f'(z)} > 0 \quad (z \in E),$$

and so $f(z) \in C(p)$.

Proof. Letting

$$g(z) = \frac{f^{(k-1)}(z)}{p(p-1)\ \cdots\ (p-k+2)},$$

then $g(z) = z^{p-k+1} + \cdots \in A(p-k+1)$ and (13) becomes

$$\left|(1-\alpha)\frac{zg'(z)}{g(z)} + \alpha\left(1+\frac{zg''(z)}{g'(z)}\right) - (p-k+1)\right| < p-k+1+\alpha \quad (z \in E)$$

Hence, by Lemma 4, we have $g(z) \in S(p-k+1)$ and

$$Re\frac{zf^{(k)}(z)}{f^{(k-1)}(z)} = Re\frac{zg'(z)}{g(z)} > 0 \quad (z \in E).$$

Applying Lemma 6 of Nunokawa [5], for $k \geqslant 3$ we get

$$\frac{f^{(k-2)}(z)}{p(p-1)\ \cdots\ (p-k+3)} = (p-k+2)\int_0^z g(t)dt \in S(p-k+2)$$

or

$$Re\frac{zf^{(k-1)}(z)}{f^{(k-2)}(z)} > 0 \quad (z \in E).$$

Making use of the same method as the above over again, we find that

$$Re\frac{zf^{(m)}(z)}{f^{(m-1)}(z)} > 0 \quad (z \in E)$$

for $m = 2, 3, \cdots, k$. This completes our proof. □

Miller and Mocanu [4, Theorem 4](see also Lemma 4 with $\alpha = p = 1$) proved that if $f(z) \in A(1)$ satisfies

$$\left|\frac{zf''(z)}{f'(z)}\right| < 2 \quad (z \in E),$$

then $f(z) \in S(1)$.

Taking $\alpha = 1$, and $k = p \geqslant 2$ in Theorem 3, we have

COROLLARY 6. *Let $p \geqslant 2$. If $f(z) \in A(p)$ satisfies*

$$\left|\frac{zf^{(p+1)}(z)}{f^{(p)}(z)}\right| < 2 \quad (z \in E),$$

then

$$Re\frac{zf''(z)}{f'(z)} > 0 \quad (z \in E),$$

which implies that $f(z) \in C(p)$.

THEOREM 4. *Let $f(z) \in A(p)$ and suppose that there exists a positive integer k for which*

$$k + Re\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} < \beta \quad (z \in E), \tag{14}$$

where $2 \leqslant k \leqslant p$ and $p < \beta \leqslant p + 1/2$. Then we have

$$Re\frac{zf''(z)}{f'(z)} > 0 \quad (z \in E),$$

and so $f(z)$ is p-valently convex in E.

Proof. Putting

$$g(z) = \frac{f^{(k-1)}(z)}{p(p-1)\ \cdots\ (p-k+2)},$$

then $g(z) \in A(p-k+1)$ and it follows from (14) that

$$\begin{aligned} Re\left\{1 + \frac{zg''(z)}{g'(z)}\right\} &= 1 + Re\frac{zf^{(k+1)}(z)}{f^{(k)}(z)} \\ &< 1 + \beta - k \leqslant (p-k+1) + \frac{1}{2} \end{aligned}$$

for $z \in E$. Therefore, from Lemma 5 we have $g(z) \in S(p-k+1)$.

The remaining part of the proof is the same as that of Theorem 3 and hence we omit it. □

REMARK 5. The above theorem improves Theorem 1 (with $2 \leqslant k \leqslant p$) and Theorem 2 of Nunokawa [6].

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