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WEAK CONVERGENCE FOR ITERATED RANDOM MAPS

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ABSTRACT. We consider a class of discrete parameter processes on a locally compact Polish space S arising from successive compositions of strictly stationary Markov random maps on S into itself. Sufficient conditions for the existence of the stationary solution and the weak convergence of the distributions of $\{\Gamma_n\Gamma_{n-1}\cdots\Gamma_0 x\}$ are given.

0. Introduction

Recently there has been considerable interest in various generalizations of autoregressive processes. Some classes of models are random coefficient models, iterated random maps models, bilinear models, stochastic difference equations, generalized autoregressive with conditional heteroskedasticity, doubly stochastic models etc. (see, e.g. [3]-[5], [9]-[15])

In this paper, we consider the iterated random maps models obtained recursively by $X_{n+1} = \Gamma_n(X_n)$, where $\{\Gamma_n : n \geq 0\}$ is a sequence of stationary Markov chains. When $\{\Gamma_n : n \geq 0\}$ are independent and identically distributed random maps , $\{X_n\}$ becomes a Markov process. But stationary Markov sequence of maps does not give rise to a Markov process in a locally compact Polish space S. Sufficient conditions ensuring the existence of strictly stationary solution for independent and identically distributed case are given in [4], [7], and [8]. It is shown by Elton[5] that if $\{\Gamma_n\}$ is a stationary sequence of Lipschitz maps on S having a.s. negative Lyapunov exponent function, $\Gamma_n\Gamma_{n-1}\cdots\Gamma_0 x$ converges in distribution to a stationary process in S. We find sufficient conditions

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under which the stationary solution exists and the weak convergence of the distribution of $\{X_n\}$ holds.

1. Main results

Let (S, ρ) be a complete, separable locally compact metric space with metric ρ. Let $C(S, S)$ be the continuous maps from S into itself, and let $Lip(S, S)$ be the Lipschitz maps from S into S. Endow $C(S, S)$ with the compact-open topology.

Consider a probability space $(\Omega, \mathcal{F}, P)$ on which are defined a sequence of stationary Markov chain $\{\Gamma_n : n \geq 0\}$ taking values on $C(S, S)$, and a random variable X_0 with values in S independent of the sequence $\{\Gamma_n : n \geq 0\}$.

In this paper, we study the convergence of the process $\{X_n\}$ generated by

$$(1.1) \qquad X_0, \quad X_n = \Gamma_{n-1}\Gamma_{n-2}\cdots\Gamma_0(X_0), \quad n \geq 1.$$

Write X_n^x for X_n with $X_0 = x$. Since $\{\Gamma_n : n \geq 0\}$ is a stationary process in a complete separable metric space, there exists $\{\tilde{\Gamma}_n : -\infty < n < \infty\}$ such that $\{\tilde{\Gamma}_n : n \geq 0\}$ have the same distribution as $\{\Gamma_n : n \geq 0\}$. For any $x \in S$, $-\infty < k < \infty$, define

$$(1.2) \qquad Y_{k,n}^x = \Gamma_{k-1}\Gamma_{k-2}\cdots\Gamma_{k-n}(x), \quad n \geq 1.$$

By stationarity of $\{\Gamma_n\}$, X_n^x has the same distribution as $Y_{k,n}^x$ for any integer k and $n \geq 1$.

For $f \in Lip(S, S)$, define

$$\|f\| = \sup_{x \neq y} \frac{\rho(f(x), f(y))}{\rho(x, y)}$$

Lipschitz norm $\|\cdot\|$ becomes a Borel measurable function.

We make the following assumptions:

Suppose there exist $x_0 \in S$, $m_0 \geq 1$ and $\lambda < 1$ such that

(A1) $\Gamma_{m_0-1}\cdots\Gamma_0$ takes values on $Lip(S, S)$,

(A2) $E_{\Gamma_0}\|\Gamma_{m_0-1}\cdots\Gamma_0\| \leq \lambda$, $\forall\omega$, and

(A3) for each $x \in S$, $\sup_{1 \leq n \leq m_0} E\rho(x_0, X_n^x) < \infty$.

LEMMA 1. *Suppose there exist* $m_0 \geq 1$ *and* $\lambda < 1$ *such that* (A1) *and* (A2) *hold. Then for each* ω,

$$E_{\Gamma_0}\|\Gamma_{nm_0-1}\Gamma_{nm_0-2}\cdots\Gamma_0\| \leq \lambda^n.$$

Proof. Note that for $f, g \in Lip(S,S)$, $\|fg\| \leq \|f\|\|g\|$. Using Markov property and stationarity of $\{\Gamma_n\}$, we have for each ω,

$$\begin{aligned}
& E_{\Gamma_0}\|\Gamma_{nm_0-1}\cdots\Gamma_0\| \\
= \ & E\left[E\left[\|\Gamma_{nm_0-1}\cdots\Gamma_0\| \mid \Gamma_{(n-1)m_0},\cdots,\Gamma_0\right] \mid \Gamma_0\right] \\
\leq \ & E\left[\|\Gamma_{(n-1)m_0-1}\cdots\Gamma_0\| \; E\left[\|\Gamma_{nm_0-1}\cdots\Gamma_{(n-1)m_0}\| \mid \Gamma_{(n-1)m_0}\right] \mid \Gamma_0\right] \\
\leq \ & \lambda \; E\left[\|\Gamma_{(n-1)m_0-1}\cdots\Gamma_0\| \mid \Gamma_0\right]
\end{aligned}$$

Repeat the same process as above, then we obtain

$$\begin{aligned}
E_{\Gamma_0}\|\Gamma_{nm_0-1}\cdots\Gamma_0\| & \leq \lambda \; E\left[\|\Gamma_{(n-1)m_0-1}\cdots\Gamma_0\| \mid \Gamma_0\right] \\
& \leq \lambda^2 \; E\left[\|\Gamma_{(n-2)m_0-1}\cdots\Gamma_0\| \mid \Gamma_0\right] \\
& \leq \cdots \leq \lambda^n.
\end{aligned} \tag{1.3}$$

□

Followings are our main theorems:

THEOREM 1. *If there exist* $x_0 \in S$, $m_0 \geq 1$, *and* $\lambda < 1$ *such that* $(A1)-(A3)$ *hold, then*

(1) *for any* $x \in S$, $Y^x_{k,n}$ *converges a.s. to* Y_k *as* $n \to \infty$ *and the distribution of* Y_k *is independent of* x.

(2) *if we take* $X_0 = Y_0$, *then* $\{Y_k : k \geq 0\}$ *is a unique stationary solution of the equation* (1.1) *and*

(3) *for any* $x \in S$, $\Gamma_n \cdots \Gamma_0(x)$ *converges in distribution to* Y_0 *as* $n \to \infty$.

Proof. (1) First we let for each $x \in S$, $K(x) = \sup_{1\leq n\leq m_0} E\rho(x_0, X^x_n)$ and $K(x) < \infty$ by (A3). For given n, find $j \geq 0$ such that $jm_0 \leq n-1 < (j+1)m_0$, then by properties of Lipschitzian norm and conditional expectation, we have

(1.4)

$$
\begin{aligned}
& E\rho(Y^{x_0}_{k,n}, Y^{x_0}_{k,n+1}) \\
\leq & E\left[\|\Gamma_{k-1}\cdots\Gamma_{k-jm_0}\|\ \rho\left(\Gamma_{k-jm_0-1}\cdots\Gamma_{k-n}(x_0), \Gamma_{k-jm_0-1}\cdots\Gamma_{k-n-1}(x_0)\right)\right] \\
= & E[E\left[\|\Gamma_{k-1}\cdots\Gamma_{k-jm_0}\|\right] \\
& \left[\rho\left(\Gamma_{k-jm_0-1}\cdots\Gamma_{k-n}(x_0), \Gamma_{k-jm_0-1}\cdots\Gamma_{k-n-1}(x_0)\right) \mid \Gamma_{k-jm_0}, \cdots, \Gamma_{k-n-1}\right]] \\
= & E[\rho\left(\Gamma_{k-jm_0-1}\cdots\Gamma_{k-n}(x_0), \Gamma_{k-jm_0-1}\cdots\Gamma_{k-n-1}(x_0)\right) \\
& E\left[\|\Gamma_{k-1}\cdots\Gamma_{k-jm_0}\| \mid \Gamma_{k-jm_0}\right]] \\
\leq & \lambda^j E\left[\rho\left(x_0, \Gamma_{k-jm_0-1}\cdots\Gamma_{k-n}(x_0)\right) + \rho\left(x_0, \Gamma_{k-jm_0-1}\cdots\Gamma_{k-n-1}(x_0)\right)\right] \\
\leq & \lambda^j(2K(x_0)).
\end{aligned}
$$

The second last inequality follows from the stationarity of $\{\Gamma_n\}$ and lemma 1.

Therefore for any integer k,

$$
\begin{aligned}
E\left[\sum_{n=1}^{\infty}\rho(Y^{x_0}_{k,n}, Y^{x_0}_{k,n+1})\right] &= \sum_{n=1}^{\infty} E\rho(Y^{x_0}_{k,n}, Y^{x_0}_{k,n+1}) \\
&\leq \sum_{n=1}^{\infty}\lambda^{[\frac{n}{m_0}]}(2K(x_0)) \\
&= \frac{2K(x_0)m_0}{1-\lambda} < \infty,
\end{aligned}
$$

where $[\frac{n}{m_0}]$ denotes the largest integer which is not greater than $\frac{n}{m_0}$. This implies that $\sum_{n=1}^{\infty}\rho(Y^{x_0}_{k,n}, Y^{x_0}_{k,n+1}) < \infty$ a.s. and hence, as $n \to \infty$, $Y^{x_0}_{k,n}$ converges a.s. to say, Y_k.

On the other hand, by the same manner as above, we have

$$
\begin{aligned}
& E\rho(Y^{x}_{k,n}, Y^{x_0}_{k,n}) \\
\leq\ & \lambda^{[\frac{n}{m_0}]} E\rho\left(\Gamma_{k-[\frac{n}{m_0}]m_0-1}\cdots\Gamma_{k-n}(x), \Gamma_{k-[\frac{n}{m_0}]m_0-1}\cdots\Gamma_{k-n}(x_0)\right) \\
\leq\ & \lambda^{[\frac{n}{m_0}]}(K(x_0)+K(x)).
\end{aligned}
$$

Hence it can be proved that for any given $x \in S$,

$$
E\left[\sum_{n=1}^{\infty}\rho(Y^{x}_{k,n}, Y^{x_0}_{k,n})\right] < \infty,
$$

which together with the convergence of $Y_{k,n}^{x_0}$ implies that $Y_{k,n}^{x}$ converges a.s. to Y_k whose distribution is independent of x.

(2) Stationarity of $\{Y_k : k \geq 0\}$ follows from that of $\{\Gamma_k : k \geq 0\}$. Also we have $Y_k = \Gamma_{k-1} \cdots \Gamma_1 \Gamma_0(Y_0)$, $k \geq 1$, and hence $\{Y_k : k \geq 0\}$ is the unique solution of (1.1).

(3) Since a.s. convergence implies the convergence in distribution, result follows from the fact that $X_n^x = \Gamma_{n-1} \cdots \Gamma_0(x)$ have the same distribution as $Y_{0,n}^x = \Gamma_{-1}\Gamma_{-2} \cdots \Gamma_{-n}(x)$. □

In next theorem, we consider the convergence of the empirical distribution of a trajectory.

THEOREM 2. *For every $x \in S$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} f(\Gamma_k \cdots \Gamma_0(x)) \to E(f(Y_0) \mid \mathcal{I}) \qquad a.s. \tag{2.5}$$

for all bounded continuous real-valued functions on S, where $\mathcal{I}$ is the σ-field of invariant events for $\{\Gamma_n\}$.

Proof. Let f be a real-valued continuous function on S with compact support. Since $\{Y_k : k \geq 0\}$ is a sequence of stationary process and $E|f(Y_0)| < \infty$, by Birkhoff's ergodic theorem,

$$\frac{1}{n} \sum_{k=0}^{n-1} f(Y_k) \to E(f(Y_0) \mid \mathcal{I}) \qquad a.s.$$

where $\mathcal{I}$ is the σ-field of invariant events for $\{\Gamma_n\}$. But $\frac{1}{n}\sum_{k=0}^{n-1} f(Y_k) = \frac{1}{n}\sum_{k=0}^{n-1} f(\Gamma_{k-1} \cdots \Gamma_0(Y_0))$ and $\rho(\Gamma_{k-1} \cdots \Gamma_0(Y_0), \Gamma_{k-1} \cdots \Gamma_0(x)) \to 0$ a.s. as $n \to \infty$, from which (1.5) holds by uniform continuity of f. For bounded continuous real-valued function f, we obtain the result from tightness of $\{\Gamma_n \cdots \Gamma_0(x)\}$ and Urysohn's lemma. □

REMARK. If $\Gamma_n \in Lip(S,S)$ and $\sup_{0 \leq n \leq m_0} \|\Gamma_n \cdots \Gamma_0\| < \infty$, then (A3) holds.

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