Bull. Korean Math. Soc. **35** (1998), No. 3, pp. 473–484

A FINITE DIFFERENCE APPROXIMATION OF A SINGULAR BOUNDARY VALUE PROBLEM

H. Y. LEE , M. R. OHM, AND J. Y. SHIN*

ABSTRACT. We consider a finite difference approximation to a singular boundary value problem arising in the study of a nonlinear circular membrane under normal pressure. It is proved that the rate of convergence is $O(h^2)$. To obtain the solution of the finite difference equation, an iterative scheme converging monotonically to the solution of the finite difference equation is introduced. And the numerical experiment of this method is given.

1. Introduction

In the study of a nonlinear circular membrane under normal pressure [3, 4], the following singular boundary value problem arises:

$$
\begin{aligned}
&-y'' - \frac{3}{x}y' - \frac{2}{y^2} = 0, \quad 0 < x < 1, \\
&y'(0) = 0, \quad \text{and} \quad y(1) = \lambda(> 0).
\end{aligned} \tag{1.1}
$$

The existence of a unique positive solution has been discussed by [2, 3, 4, 8]. Numerical solutions of this problem can be obtained by the iterative method [2] and numerical techniques [4] on the integral equation, equivalent to (1.1). Because of the singularity and the nonlinearity, difficulties are encountered if (1.1) is replaced by a finite difference equation and a numerical solution is attempted. In [7], the linearization technique to (1.1) and Gustafsson's method [6] to the linear equation

Received November 20, 1997.
1991 Mathematics Subject Classification: 65L12, 65L10.
Key words and phrases: a finite difference approximation, a singular boundary value problem, rate of convergence-$O(h^2)$.
*This work was partially supported by the research fund of Pukyong National University, 1997.

are used to avoid the above difficulities. And the error estimate of the method is given.

In this paper, we study a finite difference approximation to (1.1) which results the rate of convergence-$O(h^2)$ and which can avoid the above difficulties. The rate of convergence -$O(h^2)$ is an optimal global error when three point finite difference approximation is used. Our error estimate is better than one in [7]. To obtain the solution of the finite difference equation, we introduce an iterative scheme converging monotonically to the solution of the finite difference equation. In section 2, we consider the behaviour of the solution of (1.1) at $x = 0$ which is needed in the discretization of (1.1) near the singular point $x = 0$. In section 3, a finite difference approximation is introduced and an iterative scheme converging monotonically to the solution of the finite difference equation is given. The rate of convergence-$O(h^2)$ is established in section 4. In section 5, the rate of convergence-$O(h^2)$ is given numerically.

2. Behaviour of the solution of (1.1) at $x = 0$

To discuss the behaviour of the solution of (1.1) at $x = 0$, we begin with the following lemma whose proof is straightforward.

LEMMA 2.1. *Let $f \in C[0,\ 1]$ and $f' \in C(0,\ 1]$. If $\lim_{x \to 0+} f'(x)$ exists, then*

$$f'_+(0) = \lim_{x \to 0+} \frac{f(x) - f(0)}{x} = \lim_{x \to 0+} f'(x),$$

which implies that $f'(x)$ is continuous at $x = 0$.

It was shown in [8] that there exists a unique positive solution $Y \in C^2(0,\ 1] \cap C^1[0,\ 1]$ of (1.1). Thus we obtain the following lemma from Lemma 2.1 and the fact that

$$Y'(x) = -\frac{1}{x^3} \int_0^x \frac{2s^3}{Y^2(s)} ds.$$

LEMMA 2.2. *Let Y be the positive solution of (1.1). Then*

(1) *$Y''_+(0)$ exists and $Y''(x)$ is continuous at $x = 0$.*

(2) $Y_+^{(3)}(0)$ $(=0)$ *exists and* $Y^{(3)}(x)$ *is continuous at* $x = 0$.
(3) $Y_+^{(4)}(0)$ *exists and* $Y^{(4)}(x)$ *is continuous at* $x = 0$.

REMARK. Lemma 2.2 implies that if Y is the positive solution of (1.1) then $Y \in C^4[0,\ 1]$.

3. A finite difference approximation

Let $N \in Z^+$, $h = \frac{1}{N}$, $x_j = j \cdot h$, $y_j = y(x_j)$, $j = 0,\ 1,\ 2,\ \cdots,\ N$. Consider the following finite difference approximation:

$$
\begin{aligned}
& -8 \cdot \frac{y_1 - y_0}{h^2} - \frac{2}{y_0^2} = 0, \\
& -4 \cdot \frac{y_2 - 2y_1 + y_0}{h^2} - \frac{2}{y_1^2} = 0, \\
& -\frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{3}{x_i} \cdot \frac{y_{i+1} - y_{i-1}}{2h} - \frac{2}{y_i^2} = 0, \\
& \qquad\qquad \text{for} \quad i = 2,\ 3,\ 4,\ \cdots,\ N-1, \\
& y_N = \lambda.
\end{aligned}
\tag{3.1}
$$

Let

$L =$

$$
\begin{bmatrix}
8 & -8 & 0 & & & \cdots & 0 \\
-4 & 8 & -4 & 0 & & \cdots & 0 \\
0 & -2+\frac{3}{x_2}h & 4 & -2-\frac{3}{x_2}h & 0 & & \\
 & 0 & -2+\frac{3}{x_3}h & 4 & -2-\frac{3}{x_3}h & & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & & & 0 & -2+\frac{3}{x_{N-2}}h & 4 & -2-\frac{3}{x_{N-2}}h \\
0 & 0 & & \cdots & 0 & -2+\frac{3}{x_{N-1}}h & 4
\end{bmatrix}
$$

$$N\mathbf{y} = \left(-\frac{2h^2}{y_0^2},\ -\frac{2h^2}{y_1^2},\ -\frac{4h^2}{y_2^2},\ \cdots,\ -\frac{4h^2}{y_{N-2}^2},\ -\frac{4h^2}{y_{N-1}^2} - 2\lambda - \frac{3h}{x_{N-1}}\lambda\right)^t$$

$$\mathbf{y} = (y_0,\ y_1,\ y_2,\ \cdots,\ y_{N-1})^t,$$

where $N\mathbf{y}$ and $\mathbf{y}$ are column vectors. Now we have the nonlinear system

$$L\mathbf{y} + N\mathbf{y} = \mathbf{0}, \tag{3.2}$$

where $\mathbf{0}$ is the zero matrix. To solve the nonlinear matrix equation (3.2), we use Newton's method. So, for $m = 0,\ 1,\ 2,\ \cdots$, we have

$$\mathbf{y}^{(m+1)} = \mathbf{y}^{(m)} - \left(L + N'\mathbf{y}^{(m)}\right)^{-1} \cdot \left(L\mathbf{y}^{(m)} + N\mathbf{y}^{(m)}\right). \tag{3.3}$$

Therefore, from (3.3), we derive

$$L\mathbf{y}^{(m+1)} + \left[N'\mathbf{y}^{(m)}\right]\mathbf{y}^{(m+1)} = \left[N'\mathbf{y}^{(m)}\right]\mathbf{y}^{(m)} - N\mathbf{y}^{(m)} \tag{3.4}$$

and

(3.5)

$$\begin{aligned} L\mathbf{y}^{(m+1)} + N\mathbf{y}^{(m+1)} &= N\mathbf{y}^{(m+1)} - N\mathbf{y}^{(m)} - N'\mathbf{y}^{(m)}\left[\mathbf{y}^{(m+1)} - \mathbf{y}^{(m)}\right] \\ &= \frac{1}{2}N''\boldsymbol{\xi}^{(m)}\left(\left(y_j^{(m+1)} - y_j^{(m)}\right)^2\right), \end{aligned}$$

where $\xi_j^{(m)}$ is between $y_j^{(m+1)}$ and $y_j^{(m)}$.

LEMMA 3.1[1].

(1) *The M-matrix L is an inverse positive matrix.*

(2) *The matrix $L + N'(\mathbf{y})$ is an inverse positive matrix for any $\mathbf{y} > \mathbf{0}$.*

Proof. (1) Let D_i be the i-th leading principal minor of L. Then we obtain

$$D_1 = 8,\quad D_2 = 32,\quad D_3 = \left(2 + \frac{3h}{x_2}\right)D_2,$$

$$\cdots,\quad D_N = \left(2 + \frac{3h}{x_{N-1}}\right)D_{N-1},$$

which imply that the M-matrix L is an inverse positive matrix.
(2) It is clear that the matrix $L + N'(\mathbf{y})$ is also an inverse positive matrix for any $\mathbf{y} > \mathbf{0}$. □

LEMMA 3.2. *If* $\mathbf{u}$ *satisfies* $L\mathbf{u} + N\mathbf{u} \geq \mathbf{0}$ *and* $\mathbf{l}$ *satisfies* $L\mathbf{l} + N\mathbf{l} \leq \mathbf{0}$, *then*

$$\mathbf{l} \leq \mathbf{u},$$

where $0 < u_i$ *and* $\lambda \leq l_i$ *for* $i = 0,\ 1,\ 2,\ \cdots,\ N-1$.

Proof. From the assumptions on u and l, we have

$$\begin{aligned} \mathbf{0} &\leq L\mathbf{u} + N\mathbf{u} - L\mathbf{l} - N\mathbf{l} \\ &\leq L(\mathbf{u} - \mathbf{l}) + N(\mathbf{u} - \mathbf{l}) \\ &\leq (L + N'\boldsymbol{\xi})(\mathbf{u} - \mathbf{l}), \end{aligned}$$

where ξ_i lies between l_i and u_i. Since $L + N'\boldsymbol{\xi}$ is inverse positive, $\mathbf{u} - \mathbf{l} \geq \mathbf{0}$, which completes the proof. □

LEMMA 3.3. *If* $y_i^{(0)} = \lambda$ and $\{\mathbf{y}^{(m)}\}$ *is given by* (3.3) or (3.4), *then*

$$\mathbf{y}^{(0)} \leq \mathbf{y}^{(1)} \leq \mathbf{y}^{(2)} \leq \cdots \leq \mathbf{y}^{(m)} \leq \mathbf{u}, \quad \textit{for} \quad m = 0,\ 1,\ 2,\ \cdots,$$

where $0 < u_j$ *for* $i = 0,\ 1,\ 2,\ \cdots,\ N-1$ *and* $L\mathbf{u} + N\mathbf{u} \geq \mathbf{0}$.

Proof. It is obvious from (3.3), (3.5), and Lemma 3.2. □

LEMMA 3.4. *The matrix equation* (3.2) *has a unique solution.*

Proof. The matrix equation (3.2) has a solution from Lemma 3.3. Suppose that $\mathbf{y}$ and $\mathbf{w}$ are solutions of the matrix equation (3.2) and $\mathbf{z} = \mathbf{y} - \mathbf{w}$. Then we have

$$L\mathbf{z} + N\mathbf{y} - N\mathbf{w} = \mathbf{0}$$

So we obtain

$$(L + N'\boldsymbol{\xi})\,\mathbf{z} = \mathbf{0},$$

where ξ_i is between y_i and w_i. Since $L + N'\boldsymbol{\xi}$ is an inverse positive matrix, $\mathbf{z} = \mathbf{0}$ and hence $\mathbf{y} = \mathbf{w}$. □

4. Convergence of the difference scheme

In this section, we want to prove that the finite difference approximation (3.1) converges to the analytic solution of (1.1) as $h \to 0$ and that the rate of convergence is $O(h^2)$. First we prove the following lemma about difference operators which will be needed in establishing an error bound theorem. For this type of analysis, one is referred to [5, Section 5.5].

LEMMA 4.1. *Let $Q(x_i) = Q_i$, $E(x_i) = E_i$, be discrete functions defined on $x_0, x_1, x_2, \cdots, x_N$. Assume that there exists an $\omega > 0$ such that*

$$Q_i \le -\omega < 0, \quad i = 0, 1, 2, \cdots, N-1.$$

Set $C = \max\left(1, \frac{4}{\omega}\right)$. At the grid points $x_0, x_1, x_2, \cdots, x_{N-1}$, define the difference operator L_h by

$$L_h E_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0 E_0,$$

$$L_h E_1 = 4 \cdot \frac{E_2 - 2E_1 + E_0}{h^2} + Q_1 E_1,$$

and

$$L_h E_j = \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i, \quad i = 2, 3, \cdots, N-1.$$

Then,

$$|E_i| \le C\left[|E_N| + \max_{0 \le j < N} |L_h E_j|\right], \quad i = 0, 1, 2, \cdots, N.$$

Proof. Note that $C \ge 1$. If $\max |E_i|$ occurs for $i = N$, then

$$|E_i| \le \max_{0 \le j \le N} |E_j| \le |E_N| \le C\left[|E_N| + \max_{0 \le j < N} |E_j|\right].$$

Suppose that $\max|E_i|$ occurs for one of $i = 0, 1, 2, \cdots, N-1$. Then we have

$$\left[4 - \frac{h^2}{2}Q_0\right]|E_0| \le 4|E_1| + \frac{h^2}{2}|L_h E_0| \le 4$$

$$\cdot \sup_{0\le j\le N-1} |E_j| + \frac{h^2}{2} \cdot \sup_{0\le j\le N-1} |L_h E_j|,$$

$$\left[4 - \frac{h^2}{2}Q_1\right]|E_1| \le 2|E_2| + 2|E_0| + \frac{h^2}{2}|L_h E_1| \le 4$$

$$\cdot \sup_{0\le j\le N-1} |E_j| + \frac{h^2}{2} \cdot \sup_{0\le j\le N-1} |L_h E_j|,$$

and

$$[4 - 2h^2 Q_i]\,|E_i| \le \left|2 - \frac{3}{x_i}h\right| \cdot \sup_{0\le j\le N-1} |E_j| + \left|2 + \frac{3}{x_i}h\right| \cdot \sup_{0\le j\le N-1} |E_j|$$

$$+ 2h^2 \cdot \sup_{0\le j\le N-1} |L_h E_j|, \quad i = 2, 3, \cdots, N-1.$$

Since $\omega \le -Q_i$ and $0 \le 2 - \frac{3}{x_i}h$, we obtain

$$\left[4 + \frac{h^2}{2}\omega\right] \cdot \sup_{0\le j\le N-1} |E_j| \le 4 \cdot \sup_{0\le j\le N-1} |E_j| + 2h^2 \cdot \sup_{0\le j\le N-1} |L_h E_j|.$$

Thus

$$\sup_{0\le j\le N-1} |E_j| \le \frac{4}{\omega} \cdot \sup_{0\le j\le N-1} |L_h E_j|,$$

which completes the proof. □

THEOREM 4.2. *Let $Y = Y(x) \in C^4[0, 1]$ be the analytic solution of the boundary value problem (1.1). Let y_i, $i = 0, 1, 2, \cdots, N-1$,* *be the numerical solution of* $Ly + Ny = \mathbf{0}$ *and* $E_i = Y(x_i) - y_i$ *be the error. Then*

$$|E_i| \leq CM_4h^2$$

where

$$M_4 = \max\left|\frac{d^4Y}{dx^4}\right| \quad \text{and } C \text{ is a constant.}$$

Proof. By Mean Value Theorem and Taylor Theorem, we obtain

$$\begin{aligned}
0 &= 4Y''(x_0) + \frac{2}{Y(x_0)^2} \\
&= 8 \cdot \frac{Y(x_1) - Y(x_0)}{h^2} + \frac{2}{Y(x_0)^2} - Y^{(4)}(\xi_0) \cdot \frac{h^2}{3}, \quad x_0 < \xi_0 < x_1,
\end{aligned}$$

$$\begin{aligned}
0 &= Y''(x_1) + \frac{3}{x_1} \cdot Y'(x_1) + \frac{2}{Y(x_1)^2} \\
&= 4Y''(x_1) + 3\left(Y''(\xi_0) - Y''(x_1)\right) + \frac{2}{Y(x_1)^2} \\
&= 4 \cdot \frac{Y(x_0) - 2Y(x_1) + Y(x_2)}{h^2} - \frac{h^2}{6}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] \\
&\quad + 3Y^{(4)}(\xi_2) \cdot \xi_1(\xi_0 - x_1) + \frac{2}{Y(x_1)^2},
\end{aligned}$$

where $x_0 < \eta_0 < x_1 < \eta_1 < x_2$, $x_0 < \xi_0 < \xi_1 < x_1$, and $x_0 < \xi_2 < \xi_1$.

And for $i = 2, 3, 4, \cdots, N-1$, we obtain

$$0 = Y''(x_i) + \frac{3}{x_i} \cdot Y'(x_i) + \frac{2}{Y(x_i)^2}$$

$$= \frac{Y(x_{i-1}) - 2Y(x_i) + Y(x_{i+1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h} + \frac{2}{Y(x_i)^2}$$

$$- \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] - \frac{3}{x_i} \cdot \frac{h^2}{6}\left[Y^{(3)}(\xi_0) + Y^{(3)}(\xi_1)\right]$$

$$= \frac{Y(x_{i-1}) - 2Y(x_i) + Y(x_{i+1})}{h^2} + \frac{3}{x_i} \cdot \frac{Y(x_{i+1}) - Y(x_{i-1})}{2h} + \frac{2}{Y(x_i)^2}$$

$$+ \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right]$$

$$- \frac{3h^2}{6}\left[2Y^{(4)}(\xi_4) + Y^{(4)}(\xi_2)\frac{\xi_0 - x_i}{x_i} + Y^{(4)}(\xi_3)\frac{\xi_1 - x_i}{x_i}\right],$$

where $x_{i-1} < \eta_0 < x_i < \eta_0 < x_{i+1}$, $x_{i-1} < \xi_0 < \xi_2 < x_i < \xi_3 < \xi_1 < x_{i+1}$, and $x_0 < \xi_4 < \xi_i$.
Define

$$L_h E_0 = 8 \cdot \frac{E_1 - E_0}{h^2} + Q_0 E_0 = Y^{(4)}(\xi_0) \cdot \frac{h^2}{3},$$

$$L_h E_1 = 4 \cdot \frac{E_2 - 2E_1 + E_0}{h^2} + Q_1 E_1$$

$$= \frac{h^2}{6}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right] - 3Y^{(4)}(\xi_2) \cdot \xi_1(\xi_0 - x_1),$$

and for $i = 2, 3, \cdots, N-1$,

$$L_h E_i = \frac{E_{i+1} - 2E_i + E_{i-1}}{h^2} + \frac{3}{x_i} \cdot \frac{E_{i+1} - E_{i-1}}{2h} + Q_i E_i$$

$$= \frac{h^2}{24}\left[Y^{(4)}(\eta_0) + Y^{(4)}(\eta_1)\right]$$

$$- \frac{h^2}{2}\left[2Y^{(4)}(\xi_4) + Y^{(4)}(\xi_2)\frac{\xi_0 - x_i}{x_i} + Y^{(4)}(\xi_3)\frac{\xi_1 - x_i}{x_i}\right],$$

where $F(x,y) = \frac{2}{y^2}$, $\quad Q_i = \frac{\partial F}{\partial y}(x_i, \mu_i) = -\frac{4}{\mu_i^3} \le -\omega < 0, \quad \lambda \le \mu_i \le Y(0)$.

Let $M_4 = \max\left|\frac{d^4Y}{dx^4}\right|$. Then we obtain

$$|L_h E_0| \le \frac{h^2}{3} M_4,$$

$$|L_h E_1| \le \frac{h^2}{3} M_4 + 3h^2 M_4,$$

and for $i = 2, 3, \cdots, N-1$,

$$|L_h E_i| \le \frac{h^2}{12} M_4 + 2h^2 M_4.$$

Thus, by Lemma 4.1, we have

$$|E_i| \le C M_4 h^2, \quad \text{for} \quad i = 0, 1, 2, \cdots, N-1,$$

which completes the proof. □

5. Numerical experiment

The scheme, proposed in section 3, has been implemented on an IBM PC. In the computation, we use

$$\max_{j=0,1,\cdots,N-1}\left|y^{(k+1)}(x_j) - y^{(k)}(x_j)\right| \le \text{TOL} = 1.0 \times 10^{-7}$$

to stop the iteration when we solve the nonlinear system (3.2) by Newton's method (3.3) or (3.4). In table 1 and table 2, we report the value of $\delta_{max}(N)$ and $\delta_{min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $\lambda = 0.2,\ 0.5$, where

$$\delta_{max}(N) = \max_{j=0,1,\cdots,N-1} \frac{|y^{2N}(x_j) - y^N(x_j)|}{|y^{4N}(x_j) - y^{2N}(x_j)|},$$

$$\delta_{min}(N) = \min_{j=0,1,\cdots,N-1} \frac{|y^{2N}(x_j) - y^N(x_j)|}{|y^{4N}(x_j) - y^{2N}(x_j)|},$$

and y^N represents the solution of the nonlinear system (3.2) for the given N. From table 1 and table 2, we see numerically that Theorem 4.2 is valid.

TABLE 1. $\delta_{max}(N), \delta_{min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $\lambda = 0.2$

N	$\delta_{max}(N)$	$\delta_{min}(N)$
10	3.289144	2.834405
20	3.616159	3.399068
40	3.851348	3.769886
80	3.955196	3.930854

TABLE 2. $\delta_{max}(N), \delta_{min}(N)$ for $N = 10,\ 20,\ 40,\ 80$ and $\lambda = 0.5$

N	$\delta_{max}(N)$	$\delta_{min}(N)$
10	3.924187	3.723372
20	3.970172	3.919595
40	3.992187	3.978961
80	3.998023	3.994676

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H. Y. Lee, Department of Mathematics, Kyungsung University, Pusan 608-736, Korea
E-mail: hylee@cpd.kyungsung.ac.kr

M. R. Ohm, Department of Mathematics, Dong-Seo University, Pusan 617-716, Korea
E-mail: mrohm@kowon.dongseo.ac.kr

J. Y. Shin, Division of Mathematical Sciences, Pukyong National University, Pusan 608-737, Korea
E-mail: jyshin@dolphin.pknu.ac.kr