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ON SOME FINITE SOLUBLE GROUPS WITH ZERO DEFICIENCY

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ABSTRACT. The class of finite soluble groups with zero deficiency known to have soluble length five or six is small. In this paper we exhibit some classes of such groups.

1. Introduction

Let G be a finitely presented group, and let $\langle X|R\rangle$ be a finite presentation of G. The value $|R| - |X|$ is said to be the *deficiency* of the given presentation. The *deficiency* of G, denoted by $\mathrm{def}(G)$, is defined to be the minimum of the deficiences of all the finite presentations of G. Clearly if G is finite, then $\mathrm{def}(G) \geq 0$. A classical fact is that the rank of the Schur multiplier of G is greater than or equal to $\mathrm{def}(G)$. This goes back to Issai Schur himself [9]. B. H. Neumann asked in [7] whether a finite group with trivial multiplier has zero deficiency. Swan constructed a family of finite groups with trivial multiplier and positive deficiency, see [10]. On the other hand many examples of finite groups with zero deficiency were given by several authors. A recent account can be found in the lecture notes of D. L. Johnson [3], see also [4].

In [4] Johnson and Robertson raised the question as to whether the soluble length of a finite soluble group with zero deficiency is bounded. This question is still unanswered. All known examples of finite soluble groups with zero deficiency have soluble length less than or equal six. Only a very few of these examples are known to have soluble length five or six. Classes of such groups were given by Kenne in [5] and [6].

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In this paper we present two further classes of finite soluble groups $G_i(m), i = 1, 2$, of soluble length six with 2-generator 2-relation presentations. We shall show that a subclass of $G_2(m)$ is the class of groups discussed in [5]. Our construction is similar to that of Kenne and employs the minimal soluble groups of soluble length six [1], as a starting point. Using a similar method we shall proceed to construct two new classes of finite soluble groups $H_i(m)$, $i = 0, 1$, of soluble length five having zero deficiencies. Finally we present an example of finite soluble group with trivial multiplier and positive deficiency whose soluble length is five.

We use the notation C_n for the cyclic group of order n and the notation $[x, y]$ for $x^{-1}y^{-1}xy (= x^{-1}x^y)$.

2. The Groups $G_i(m)$

We begin with a definition given in [1].

DEFINITION 2.1. A finite soluble group is said to have *minimal order* amongst the soluble groups of soluble length d, if every soluble group H of soluble length d has order at least that of G. We denote by $\mathcal{MO}(d)$ the set of all soluble groups of soluble length d with minimal order.

It has been shown in [1] that there are precisely three groups in $\mathcal{MO}(6)$, namely the extensions of the extraspecial group N of order 3^3 and exponent 3 by $GL(2,3)$, where $GL(2,3)$ acts naturally on N/N'. The following 2-generator 2-relation presentations are given for the three members of $\mathcal{MO}(6)$ (see [1]).

$$\begin{aligned}
G_0 &= \langle x, y \mid xy(x^{-1}y)^2xy^{-2} = x^2yx^{-1}y^{-3}xy^2 = 1\rangle, \\
G_1 &= \langle x, y \mid (xy)^2xy^{-2}x^{-1}y = x^2yx^{-2}yx^2y^{-2} = 1\rangle, \\
G_2 &= \langle x, y \mid (xy)^2y^{-6} = x^4y^{-1}xy^{-9}x^{-1}y = 1\rangle.
\end{aligned}$$

It is seen readily that $G_i/G_i' \cong C_2$ $(0 \leq i \leq 2)$. So that, using the Schur-Küneth formula [9], $G_i \times C_{2k-1}$, where $k \in \mathbb{N}$, has trivial multiplier. Clearly $G_i \times C_{2k-1}$ is a soluble group of soluble length 6. Now the following question arises.

QUESTION. Does $G_i \times C_{2k-1}$ have zero deficiency ($i = 0, 1, 2$, and $k \in \mathbb{N}$)?

If the answer is affirmative, we have constructed classes of finite soluble groups of soluble length 6 and zero deficiency.

Kenne in [5] showed that $G_2 \times C_{|k+8|}$, where $k \equiv 3 \pmod 6$, has the following 2-generator 2-relation presentation

$$G(k) = \langle a, b \mid (ab)^2 = b^6,\ a^4b^{-1}ab^ka^{-1}b = 1\rangle.$$

We shall below prove that $G_i \times C_{2t-1}$ ($i = 1, 2$, and $t \in \mathbb{N}$) has zero deficiency. We also see that $G_0 \times C_{|8t-3|}$ where $t \in \mathbb{Z}$, has a 2-generator 2-relation presentation.

Taking $u = y^{-1}x$, $v = (xy^2)^{-1}$ in G_1 and $u = y^{-1}x$, $v = yxy^4$ in G_2, we observe that the pair (u, v) generate G_i ($i = 1, 2$) by using GAP [8]. Now the following presentations for G_1 and G_2 are found on the new generators

$$G_1 = \langle u, v \mid u^2 = vu^{-2}v,\ v^u = u^{v^3}\rangle,$$
$$G_2 = \langle u, v \mid u^2 = vu^{-2}v,\ v^{u^{-2}} = v^{u^v}\rangle.$$

These presentations motivate our definition of the classes of groups $G_i(m)$ ($i = 1, 2$, and $m \in \mathbb{Z}$) defined by

$$G_1(m) = \langle u, v \mid u^2 = vu^mv,\ v^u = u^{v^3}\rangle,$$
$$G_2(m) = \langle u, v \mid u^2 = vu^mv,\ v^{u^{-2}} = v^{u^v}\rangle.$$

We shall show that if $m \equiv -2 \pmod 8$, then $G_i(m) \cong G_i \times C_{|m/2|}$ ($i = 1, 2$). Therefore, $G_i(m)$ being a finite soluble group of soluble length 6, will have zero deficiency.

Before proceeding further we note that u^{m+2} is central in $G_i(m)$ and that $G_i(m)/\langle u^{m+2}\rangle \cong G_i$ ($i = 1, 2$) whenever $m \equiv -2 \pmod 8$. We also observe that $G_i(m)/G_i'(m) \cong C_{|m|}$. So $G_i(m)$ is finite using a result of Schur which states that if a central subgroup of a group K has finite index, then the derived group of K is finite. On the other hand it is obvious that $G_i(m)$ has soluble length at least six.

The proof of the result depends on the following two lemmas.

LEMMA 2.2. *Let $H = \langle v^{-1}u, u^t \rangle$ where t is a positive integer dividing m, then $H \lhd G_i(m)$ and $|G_i(m) : H| = t$ $(i = 1, 2)$.*

Proof. Since $uv^{-1} = u^{-1}vu^m$, we see that $uv^{-1} \in H$, so $v(v^{-1}u)v^{-1} \in H$. Also it is clear that $u(v^{-1}u)u^{-1} \in H$. Now by considering $vu^tv^{-1} = vu^{-1}u^tuv^{-1}$, we find that $vu^tv^{-1} \in H$. This proves that $H \lhd G_i(m)$. The second part is now obvious. □

LEMMA 2.3. *In $G_i(m)$ $(i = 1, 2)$, if m is even then the following relation holds*

$$u^{4m} = 1.$$

Proof. Let $a = v^{-1}u$, $b = u^2$. It is straightforward to check that $[a, b^{(m+2)/2}] = 1$. Also we see that the relations

$$(R_1) \qquad a^{baba^{-1}} = a^{-2}b^{m/2}, \quad (ab)^{a^{-1}(ba)^2} = a^{-2}b^{m+1}$$

hold in $G_1(m)$, and similarly the relations

$$(R_2) \qquad (a^3)^{ba} = b^m, \quad (ab)^3 = (b^{\frac{m}{2}+3})^a$$

hold in $G_2(m)$. Taking $c = b^{m/2}$ the relations (R_1) and (R_2) reduce to the relations

$$(S_1) \qquad a^{c^{-1}ac^{-1}a^{-1}} = a^{-2}c, \quad ca^{-1}cac = ac^{-1}aca^{-1}ca^2$$

and the relations

$$(S_2) \qquad a^3 = (c^2)^{a^{-1}c}, \quad c^2a^2c^{-1}ac^{-1} = aca^{-1},$$

respectively. Now introducting the group

$$L = \langle a, c \mid \text{ either } \ (S_1) \ \text{ or } \ (S_2) \rangle,$$

we see that L has order 648 and the relation $c^4 = 1$ holds in L, by using GAP. Therefore, $u^{4m} = b^{2m} = c^4 = 1$. □

THEOREM 2.4. *If* $m \equiv -2 \pmod 8$, *then* $G_i(m) \cong G_i \times C_{|m/2|}$ $(i = 1, 2)$.

Proof. Let $H = \langle v^{-1}u, u^{m/2}\rangle$, and let $K = \langle u^{m+2}\rangle$. Clearly $G_i(m) = HK$. Now, suppose that $x \in H \cap K$. So $x = u^{(m+2)l}$ for some integer l. Therefore $u^d \in H$, where $d = gcd(l, \frac{m}{2})$, as $\frac{m}{2}$ is odd. Using Lemma 2.2, $d = \frac{m}{2}$ and hence $l = \frac{m}{2}q$ for some interger q. Now Lemma 2.3 guarantees that $x = 1$, that is, $G_i(m) \cong H \times K$. But $H \cong G_i(m)/K \cong G_i$ as required. □

REMARK 1. It was found that the group $G(n)$ defined by

$$G(n) = \langle a, b \mid a^2 = b^n, \quad (ba)^{b^{-2}} = [b, aba]\rangle,$$

where $n \equiv -8 \pmod{24}$ is, in fact, isomorphic to $G_2 \times C_{|\frac{n+2}{2}|}$. This together with the Kenne's presentation for $G_2 \times C_{|k+8|}$, where $k \equiv 3 \pmod 6$, gives a second proof for the group $G_2 \times C_{2t-1}$ to have zero deficiency for all positive integers t.

REMARK 2. Having failed to find a presentation for $G_0 \times C_{2k-1}$, $k \in \mathbb{N}$, with the same number of generators and relations, we were able to show that the 2-generator 2- relation group defined by

$$\langle a, b \mid a^2 = b^8, \quad (ba)^2 b^m a^{-1} b^{-1} a b^{-2} a = ab^2ab\rangle,$$

where $m \equiv -3 \pmod 8$, is isomorphic to $G_0 \times C_{|m|}$.

We note that it order to establish the above isomorphisms we may appeal to the same method that was used to prove Theorem 2.4.

3. The Groups $H_i(m)$

Adopting the same strategy we tried to construct a class of finite soluble groups of soluble length five using the only member of $\mathcal{MO}(5)$, namely the group M defined by a 2-generator 2-relation presentation

$$\langle x, y | x^2y^3 = (xy)^3(xy^{-1}xy)^2x^{-1}yxy^{-1}x^{-1}y(yx)^{-3}x^{-1}y = 1\rangle,$$

see [1]. (The above presentation of M is due to the author [2].) We observed that $M \times C_3$ has trivial multiplier but positive deficiency. This example was generalized by C. I. Wotherspoon to obtain an infinite class of soluble groups with trivial multiplier and positive deficiency, see [12] for a detailed treatment. The example also shows that the direct product of two groups with zero deficiency need not to have zero deficiency.

Accordingly we proceed to consider the three known soluble groups of order 648 with soluble length five. These are presented by the following 2-generator 2-relation presentations

$$\begin{aligned} H_0 &= \langle x, y \mid xy^2x^{-1}y^2xy^{-1} = xyxy^{-1}x^{-1}y^{-2}x^{-1}y^{-1} = 1\rangle, \\ H_1 &= \langle x, y \mid (xy)^2x^{-2}y^{-5} = xy^2x^{-1}y^3x^{-1}y^2 = 1\rangle, \\ H_2 &= \langle x, y \mid x^3y(x^{-1}y)^2 = x^2yxy(xyx)^{-1}yxy = 1\rangle, \end{aligned}$$

see [11].

Here we see that $H_i/H_i' \cong C_3$ $(0 \le i \le 2)$. Again it is obvious that $H_i \times C_m$ has trivial multiplier when $gcd(m, 3) = 1$. Now one may ask whether $H_i \times C_m$ has zero deficiency when $gcd(m, 3) = 1$. Kenne in [6] showed that this is the case whenever $i = 2$. We assert that the answer remains true in two other cases. So we have the following theorem.

THEOREM 3.1. *Define the groups $H_i(m)$ $(i = 0, 1)$ by*

$$\begin{aligned} H_0(m) &= \langle u, v \mid (uv)^3 = v^3, \quad v^{u^2} = v^m u^{-2} v\rangle, \\ H_1(m) &= \langle u, v \mid (uv)^4 = v^3, \quad u^{v^{-1}u^2} = v^m u^3 v^{-1}\rangle, \end{aligned}$$

where $m \equiv -1 \pmod 3$. Then $H_0(m) = H_0 \times C_{|m|}$, and $H_1(m) = H_1 \times C_{|m-1|}$. Hence, $H_i(m)$ is a finite soluble group of soluble length 5 *having zero deficiency.*

The proof is similar to that of Theorem 2.4, so we omit the detail.

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