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NOTES ON EXTRINSIC SPHERES

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ABSTRACT. The main purpose of this paper is to give a characterization of a totally geodesic Kaehler submanifold M of a Kaehler manifold $\widetilde{M}$ by observing the extrinsic shape of *particular* circles of the submanifold M.

0. Introduction

In submanifold theory totally geodesic submanifolds are the simplest examples. They are characterized by the condition that *all* geodesics of the submanifold are geodesics in the ambient space. Here we study totally umbilic submanifolds with parallel mean curvature vector. These submanifolds M's have a property that *all* circles of the submanifold M are circles in the ambient space $\widetilde{M}$. They are usually called *extrinsic spheres* of $\widetilde{M}$. Nomizu and Yano [5] proved that M is an extrinsic sphere of $\widetilde{M}$ if and only if there exists some positive constant k and all circles of curvature k on M are circles on $\widetilde{M}$.

In this paper we first improve this result. In the latter half of this paper we pay particular attention to extrinsic *Kaehler* submanifolds M's of an arbitrary Kaehler manifold $\widetilde{M}$. Needless to say, these extrinsic submanifolds M's are necessarily minimal so that they are totally geodesic in $\widetilde{M}$. Motivated by this fact, we shall provide a characterization of a totally geodesic Kaehler submanifold M in a Kaehler manifold $\widetilde{M}$ by observing the extrinsic shape of *particular* circles of the submanifold M (cf. Theorem).

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1. Holomorphic circles

We first recall the definition of circles. A smooth curve $\gamma = \gamma(s)$, parametrized by arclength s, in a Riemannian manifold M is called a *circle of curvature* k if there exist a field $Y = Y(s)$ of unit vectors along γ and a positive constant k satisfying

$$(1.1) \qquad \begin{cases} \nabla_{\dot\gamma}\dot\gamma = kY \\ \nabla_{\dot\gamma}Y = -k\dot\gamma, \end{cases}$$

where $\dot\gamma$ denotes the unit tangent vector of γ and ∇ the covariant differentiation. As a matter of course any Riemannian manifold has many circles. In fact, for an arbitrary point x, an arbitrary orthonormal pair (u, v) of vectors at x and an arbitrary positive number k, there exists locally a unique circle $\gamma = \gamma(s)$ with initial condition $\gamma(0) = x$, $\dot\gamma(0) = u$ and $Y(0) = v$.

We next consider some particular circles in a Kaehler manifold. Let γ be a circle in a Kaehler manifold M (with metric $\langle\ ,\ \rangle$ and complex structure J). Then we see from (1.1) that $\langle\dot\gamma, JY\rangle$ is constant along γ. Hence it makes sense to define a *holomorphic circle* in M as a circle γ satisfying that $\dot\gamma$ and Y span a holomorphic plane, that is, $Y = J\dot\gamma$ or $Y = -J\dot\gamma$. So, if γ is a holomorphic circle, then (1.1) reduces to

$$(1.2) \qquad \nabla_{\dot\gamma}\dot\gamma = kJ\dot\gamma \quad \text{or} \quad \nabla_{\dot\gamma}\dot\gamma = -kJ\dot\gamma.$$

We can interpret such circles in terms of physics (see [1]). Holomorphic circles in Kaehler Geometry may play a similar role of geodesics in Real Riemannian Geometry.

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2. Extrinsic spheres

Our aim here is to prove the following which is an improvement of the well-known result of Nomizu and Yano.

PROPOSITION. *Let $M^n, n \geqq 2$, be a connected submanifold of a Riemannian manifold $\widetilde{M}^m$. Then M^n is an extrinsic sphere in $\widetilde{M}^m$ if*

and only if for some positive constant k there exists an orthonormal basis $\{v_1, \ldots, v_n\}$ at each point p of M^n satisfying the following three conditions:
(i) *Every circle $\gamma_{ij}(s)$ of curvature k in M^n with $\gamma_{ij}(0) = p$, $\dot\gamma_{ij}(0) = v_i$ and $(\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}(s))_{s=0} = kv_j$ $(1 \leqq i \neq j \leqq n)$ is a circle in $\widetilde{M}^m$.*
(ii) *Every circle $\tau_{ij}(s)$ of curvature k in M^n with $\tau_{ij}(0) = p$, $\dot\tau_{ij}(0) = v_i$ and $(\nabla_{\dot\tau_{ij}}\dot\tau_{ij}(s))_{s=0} = -kv_j$ $(1 \leqq i \neq j \leqq n)$ is a circle in $\widetilde{M}^m$.*
(iii) *Every circle $\delta_{ij}(s)$ of curvature k in M^n with $\delta_{ij}(0) = p$, $\dot\delta_{ij}(0) = (v_i + v_j)/\sqrt{2}$ and $(\nabla_{\dot\delta_{ij}}\dot\delta_{ij}(s))_{s=0} = k(v_i - v_j)/\sqrt{2}$ $(1 \leqq i \neq j \leqq n)$ is a circle in $\widetilde{M}^m$.*

Proof. Let $\gamma_{ij} = \gamma_{ij}(s)$ be a circle in M^n satisfying condition (i). Then from (1.1) we know that γ_{ij} yields

$$\nabla^2_{\dot\gamma_{ij}}\dot\gamma_{ij} + \langle\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}, \nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}\rangle\dot\gamma_{ij} = 0. \tag{2.1}$$

By hypothesis γ_{ij} is a circle in the ambient space $\widetilde{M}^m$, so that it shows

$$\widetilde\nabla^2_{\dot\gamma_{ij}}\dot\gamma_{ij} + \langle\widetilde\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}, \widetilde\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}\rangle\dot\gamma_{ij} = 0. \tag{2.2}$$

Equations (2.1) and (2.2), together with the formulae of Gauss $\widetilde\nabla_X Z = \nabla_X Z + \sigma(X, Z)$ and Weingarten $\widetilde\nabla_X\xi = D_X\xi - A_\xi X$, yield

$$A_{\sigma(\dot\gamma_{ij},\dot\gamma_{ij})}\dot\gamma_{ij} = \langle\sigma(\dot\gamma_{ij}, \dot\gamma_{ij}), \sigma(\dot\gamma_{ij}, \dot\gamma_{ij})\rangle\dot\gamma_{ij} \tag{2.3}$$

and

$$\sigma(\dot\gamma_{ij}, \nabla_{\dot\gamma_{ij}}\dot\gamma_{ij}) + D_{\dot\gamma_{ij}}(\sigma(\dot\gamma_{ij}, \dot\gamma_{ij})) = 0. \tag{2.4}$$

At $s = 0$ in (2.4), we get, noting $(\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij})(0) = kv_j$,

$$\sigma(v_i, v_j) = (-1/k)D_{v_i}(\sigma(v_i, v_i)) \quad (1 \leqq i \neq j \leqq n). \tag{2.5}$$

Similarly, condition (ii) in Proposition tells us

$$\sigma(v_i, v_j) = (1/k)D_{v_i}(\sigma(v_i, v_i)) \quad (1 \leqq i \neq j \leqq n). \tag{2.6}$$

Hence, Equations (2.5) and (2.6) show

$$\sigma(v_i, v_j) = 0 \quad (1 \leqq i \neq j \leqq n). \tag{2.7}$$

Exchanging i for j in (iii) in Proposition, we have similarly

$$\sigma\left(\frac{v_i + v_j}{\sqrt{2}}, \frac{v_i - v_j}{\sqrt{2}}\right) = 0$$

so that

$$\sigma(v_i, v_i) = \sigma(v_j, v_j) \quad (1 \leqq i \neq j \leqq n). \tag{2.8}$$

From (2.7) and (2.8) we know that M^n is totally umbilic in $\widetilde{M}^m$, since p is any point of M^n. This, combined with Equation (2.4), implies

$$D_{\dot{\gamma}_{ij}}(\sigma(\dot{\gamma}_{ij}, \dot{\gamma}_{ij})) = 0. \tag{2.9}$$

Here, again by using the fact that M^n is totally umbilic in $\widetilde{M}^m$, at $s = 0$ in (2.9) we find that $D_{v_i}\mathfrak{h} = 0$ $(1 \leqq i \leqq n)$ at any point p of M, where $\mathfrak{h}$ is the mean curvature vector of M in $\widetilde{M}$. This implies that the mean curvature vector of M^n in $\widetilde{M}^m$ is parallel.

The converse is trivial from [5]. □

3. Main result

THEOREM. *Let M be an n-dimensional Kaehler submanifold of an $(n+p)$-dimensional Kaehler manifold $\widetilde{M}$ (with complex structure J). Then M is totally geodesic in $\widetilde{M}$ if and only if for some positive constant k there exists such an orthonormal basis $\{v_1, \cdots, v_n, Jv_1, \cdots, Jv_n\}$ at each point p of M that all holomorphic circles γ_{ij} of curvature k in M through p satisfying that the initial vector $\dot{\gamma}_{ij}(0)$ is in the direction $v_i + v_j$ $(1 \leq i \leq j \leq n)$ are circles in $\widetilde{M}$.*

Proof. We denote by σ the second fundamental form of the submanifold M in $\widetilde{M}$. Then it satisfies (cf. [6])

$$\sigma(JX,Y) = \sigma(X,JY) = J\sigma(X,Y). \tag{3.1}$$

Let γ_{ii} $(1 \leq i \leq n)$ be circles of M satisfying $\nabla_{\dot\gamma_{ii}}\dot\gamma_{ii} = kJ\dot\gamma_{ii}$ with $\gamma_{ii}(0) = p$ and $\dot\gamma_{ii}(0) = v_i$. By hypothesis they satisfy

$$\widetilde{\nabla}^2_{\dot\gamma_{ii}}\dot\gamma_{ii} = -k_i^2\dot\gamma_{ii} \tag{3.2}$$

for some positive constant k_i. On the other hand, we have

$$\begin{aligned}\widetilde{\nabla}_{\dot\gamma_{ii}}\dot\gamma_{ii} &= \nabla_{\dot\gamma_{ii}}\dot\gamma_{ii} + \sigma(\dot\gamma_{ii},\dot\gamma_{ii})\\ &= kJ\dot\gamma_{ii} + \sigma(\dot\gamma_{ii},\dot\gamma_{ii}).\end{aligned}$$

Note that $\widetilde{\nabla}J = 0$. Hence

$$\widetilde{\nabla}^2_{\dot\gamma_{ii}}\dot\gamma_{ii} = kJ(\nabla_{\dot\gamma_{ii}}\dot\gamma_{ii} + \sigma(\dot\gamma_{ii},\dot\gamma_{ii})) - A_{\sigma(\dot\gamma_{ii},\dot\gamma_{ii})}\dot\gamma_{ii} + D_{\dot\gamma_{ii}}(\sigma(\dot\gamma_{ii},\dot\gamma_{ii})). \tag{3.3}$$

Comparing the normal component of (3.2) and (3.3), at $s = 0$ we find

$$D_{v_i}(\sigma(v_i,v_i)) + kJ\cdot\sigma(v_i,v_i) = 0. \tag{3.4}$$

Applying the same discussion as above to the circle γ_{ii} in M satisfying $\nabla_{\dot\gamma_{ii}}\dot\gamma_{ii} = -kJ\dot\gamma_{ii}$ with $\gamma_{ii}(0) = p$ and $\dot\gamma_{ii}(0) = v_i$, we get

$$D_{v_i}(\sigma(v_i,v_i)) - kJ\cdot\sigma(v_i,v_i) = 0. \tag{3.5}$$

It follows from (3.4) and (3.5) that

$$\sigma(v_i,v_i) = 0\ (1 \leq i \leq n). \tag{3.6}$$

Similarly, considering the circles γ_{ij} $(1 \leq i < j \leq n)$ in M satisfying $\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij} = kJ\dot\gamma_{ij}$ or $\nabla_{\dot\gamma_{ij}}\dot\gamma_{ij} = -kJ\dot\gamma_{ij}$ with $\gamma_{ij}(0) = p$ and $\dot\gamma_{ij}(0) = (v_i + v_j)/\sqrt{2}$, we obtain

$$\sigma\left(\frac{v_i+v_j}{\sqrt{2}}, \frac{v_i+v_j}{\sqrt{2}}\right) = 0\ (1 \leq i < j \leq n). \tag{3.7}$$

It follows from (3.1), (3.6) and (3.7) that the submanifold M is geodesic at the point p. Thus we get the conclusion. □

Finally we recall geometric properties of holomorphic circles on some Kaehler manifolds (for details, see [2, 3, 4]).

(1) An n-dimensional complex projective space $\mathbb{C}P^n(c)$ of constant holomorphic sectional curvature c can be imbedded into $\mathbb{C}P^{n+p}(c)$ as a totally geodesic Kaehler submanifold. Every holomorphic circle in $\mathbb{C}P^n(c)$ is a closed curve with length $\frac{2\pi}{\sqrt{k^2+c}}$ which lies on the totally geodesic Kaehler submanifold $\mathbb{C}P^1(c)$ of $\mathbb{C}P^n(c)$.

(2) An n-dimensional complex hyperbolic space $\mathbb{C}H^n(c)$ of constant holomorphic sectional curvature c can be imbedded into $\mathbb{C}H^{n+p}(c)$ as a totally geodesic Kaehler submanifold. A holomorphic circle γ (of curvature k) in $\mathbb{C}H^n(c)$ is not necessarily closed. The holomorphic circle γ is closed if and only if $k > \sqrt{|c|}$. Its length is $\frac{2\pi}{\sqrt{k^2+c}}$.

(3) $\mathbb{C}P^1(c) \times \mathbb{C}P^1(c)$ can be imbedded into a complex Grassmannian manifold $G_2(\mathbb{C}^4)$ (of 2-dimensional complex linear subspaces in $\mathbb{C}^4$) with maximal sectional curvature c as a totally geodesic Kaehler submanifold. A holomorphic circle $\gamma = \gamma(s)$ (of curvature k with initial unit vector $\dot\gamma(0) = (X_1, X_2)$) in $\mathbb{C}P^1(c) \times \mathbb{C}P^1(c)$ is not necessarily closed. When $X_1 = 0$ or $X_2 = 0$, the holomorphic circle γ is a closed curve with length $\frac{2\pi}{\sqrt{k^2+c}}$. When $X_1 \neq 0$ and $X_2 \neq 0$, the holomorphic circle γ is closed if and only if $\sqrt{\frac{k^2+c\|X_2\|^2}{k^2+c\|X_1\|^2}}$ is rational. Its length is the least common multiple of $\frac{2\pi}{\sqrt{k^2+c\|X_1\|^2}}$ and $\frac{2\pi}{\sqrt{k^2+c\|X_2\|^2}}$. Here for two positive real numbers α, β, when the ratio $\frac{\alpha}{\beta}$ is rational, we define the least common multiple of α and β as the minimal value of the set $\{\alpha n |\ n = 1, 2, 3, \cdots\} \cap \{\beta n |\ n = 1, 2, 3, \cdots\}$.

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