Bull. Korean Math. Soc. **35** (1998), No. 3, pp. 503–514

A CHARACTERIZATION OF HOROSPHERES AND GEODESIC HYPERSPHERES IN A COMPLEX HYPERBOLIC SPACE IN TERMS OF RICCI TENSORS

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ABSTRACT. We want to treat this problem for real hypersurfaces in a complex hyperbolic space $H_n(C)$. Thus it seems to be natural to consider some problems concerned with the estimation of the Ricci tensor for real hypersurfaces in $H_n(C)$. In this paper we will find a new tensorial formula concerned with the Ricci tensor and give it a characterization of horospheres and geodesic hyperspheres in a complex hyperbolic space $H_n(C)$.

0. Introduction

Let $H_n(C)$ be an n-dimensional complex hyperbolic space equipped with the Bergman metric of constant holomorphic sectional curvature -4, and let us denote by M a real hypersurface of $H_n(C)$. Then naturally M admits so called an induced almost contact metric structure (ϕ, ξ, η, g) induced from the almost complex structure J of $H_n(C)$.

Recently many differential geometers ([2], [5], [6], [7], [13]) have studied several characterizations of homogeneous real hypersurfaces in $H_n(C)$ which are said to be of type A_0, A_1, A_2 and B, introduced as model hypersurfaces in the works of Berndt [2], [3], Montiel [13], and Montiel and Romero [14]. In particular, by using the notion of the tube in Cecil and Ryan [4], Montiel [13] also classified the real hypersurfaces of a complex hyperbolic space with at most two distinct principal curvatures.

Received October 17, 1997.

1991 Mathematics Subject Classification: Primary 53C40; Secondary 53C15.

Key words and phrases: horosphere, geodesic hypersphere, estimation of the Ricci tensor.

This paper was supported by the Korean Science and Engineering Foundation, Project No. 971-0104-018-1.

Also Berndt [2] classified all real hypersurfaces with constant principal curvature of $H_n(C)$ under the condition that the structure vector field ξ is principal. Namely he proved the following.

THEOREM A. *Let M be a connected real hypersurface of $H_n(C)$ $(n\geq 2)$. Then M has constant principal curvatures and ξ is principal if and only if M is locally congruent to one of the following*

(A_0) *a horosphere in $H_n(C)$,*

(A_1) *a geodesic hypersphere or a tube over a complex hyperplane $H_{n-1}(C)$.*

(A_2) *a tube over a totally geodesic submanifold $H_k(C)$ for $k=1,...,n-2$.*

(B) *a tube over a totally real hyperbolic space $H_n(R)$.*

As can be seen from the example above real hypersurface of type A_0 is said to be a *horosphere*, which has no focal points and which is congruent to all its parallel hypersurfaces, and that of type A_1 is said to be a *geodesic hypersphere.* Garay and Romero [6] constructed an isometric embedding of $H_n(C)$ into a pseudo-Euclidean space and characterize the *horosphere* by computing the Laplacian of the mean curvature vector field of M in a pseudo-Eclidean space.

On the other hand, if we use the equation of Codazzi, we know that there does not exist a real hypersurface in $H_n(C)$ with the parallel second fundamental tensor. From this point of view, Chen, Ludden and Montiel [5] calculated the norm of the derivative of the second fundamental tensor for the real hypersurfaces in $H_n(C)$ and showed that it is estimated by $\|\nabla A\|^2\geq 4(n-1)$, where the equality holds if and only if M is of type A_0, A_1 and A_2.

Moreover, in a paper [7] Ki, Nakagawa and Suh proved that there does not exist a real hypersurface of non-flat complex space forms $M_n(c)$ with harmonic Weyl tensor. So it follows that there does not exist a Einstein or a real hypersurface with parallel Ricci tensor in non-flat complex space forms. From this fact in a real hypersurface of a complex projective space $P_n(C)$ Kimura and Maeda [11] estimated the norm of the derivative of the Ricci tensor and obtained a characterization of a geodesic hypersphere in $P_n(C)$.

Now we want to treat this problem for real hypersurfaces in a complex hyperbolic space $H_n(C)$. Thus it seems to be natural to consider some problems concerned with the estimation of the Ricci tensor for real hypersurfaces in $H_n(C)$. In this paper we will find a new tensorial formula concerned with the Ricci tensor and give it a characterization of horospheres and geodesic hyperspheres in a complex hyperbolic space $H_n(C)$ by the following.

THEOREM 1. *Let M be a real hypersurface in $H_n(C)$. Then the Ricci tensor S of M satisfies*

$$(\nabla_X S)Y = c\{g(\phi X, Y)\xi + \eta(Y)\phi X\} \tag{0.1}$$

for a non-zero constant c if and only if M is locally congruent to one of type A_0 and A_1.

Finally, as an application of this characterization we will estimate the norm of the covariant derivative of the Ricci tensor for these types by the following

THEOREM 2. *Let M be a real hypersurface with constant mean curvature in $H_n(C)$. Then the following inequality holds*

$$\|\nabla S\|^2 \geq \frac{4n}{n-1}(h-\alpha)\{n(h-\alpha) + Tr(\phi A \nabla_\xi A)\}, \tag{0.2}$$

where the above equality holds if and only if M is locally congruent to one of type A_0 and A_1, provided that $\alpha = \eta(A\xi)$ is constant.

1. Preliminaries

Let M be an orientable real hypersurface of $H_n(C)$ and let N be a unit normal vector field on M. The Riemannian connections $\tilde{\nabla}$ in $H_n(C)$ and ∇ in M are related by the following formulas for any vector fields X and Y on M:

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \tag{1.1}$$

$$\tilde{\nabla}_X N = -AX, \tag{1.2}$$

where g denotes the Riemannian metric of M induced from the Bergman metric G of $H_n(C)$ and A is the shape operator of M in $H_n(C)$.

An eigenvector X of the shape operator A is called a principal curvature vector. Also an eigenvalue λ of the shape operator A is called a principal curvature. In what follows, we denote by V_λ the eigenspace of A associated with eigenvalue λ. It is known that M has an almost contact metric structure induced from the complex structure J on $H_n(C)$, that is, we define a tensor field ϕ of type (1,1), a vector field ξ and a 1-form η on M by $g(\phi X, Y) = G(JX, Y)$ and $g(\xi, X) = \eta(X) = G(JX, N)$. Then we have

$$\phi^2 X = -X + \eta(X)\xi, g(\xi,\xi) = 1, \phi\xi = 0. \tag{1.3}$$

It follows from (1.1) that

$$(\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi, \tag{1.4}$$

$$\nabla_X \xi = \phi AX. \tag{1.5}$$

Let $\tilde{R}$ and R be the curvature tensors of $H_n(C)$ and M, respectively. Since the curvature tensor $\tilde{R}$ has a nice form, we have the following Gauss and Codazzi equations

$$\begin{aligned} g(R(X,Y)Z,W) = &- g(Y,Z)g(X,W) \\ &+ g(X,Z)g(Y,W) - g(\phi Y, Z)g(\phi X, W) \\ &+ g(\phi X, Z)g(\phi Y, W) + 2g(\phi X, Y)g(\phi Z, W) \\ &+ g(AY,Z)g(AX,W) - g(AX,Z)g(AY,W), \end{aligned} \tag{1.6}$$

$$(\nabla_X A)Y - (\nabla_Y A)X = -\eta(X)\phi Y + \eta(Y)\phi X + 2g(\phi X, Y)\xi, \tag{1.7}$$

where R denotes the Riemannian curvature tensor of M and $\nabla_X A$ is the covariant derivative of the shape operator A with respect to X.

The Ricci tensor S' of M is the tensor of type (0,2) which is given by $S'(X,Y) : trZ{\rightarrow}R(Z,X)Y$. Also it may be regarded as the tensor of type (1,1) and denote by $S : TM{\rightarrow}TM$; it satisfies $S'(X,Y) = g(SX,Y)$. From (1.6) we see that the Ricci tensor S of M is given by

$$S = -(2n+1)I + 3\eta\otimes\xi + hA - A^2, \tag{1.8}$$

where we have put $h = trA$. The covariant derivative of (1.5) are given as follows

(1.9)

$$(\nabla_X S)Y$$
$$= 3(\nabla_X\eta)(Y) + 3\eta(Y)\nabla_X\xi + (Xh)AY + h(\nabla_X A)Y - (\nabla_X A^2)Y.$$

2. Proof of Theorem 1

Let M be a real hypersurface in a complex hyperbolic space $H_n(C)$. Firstly let us suppose that M is a horosphere in $H_n(C)$ which is said to be of type A_0. Then its Weingarten map A is given by

$$AX = X + \eta(X)\xi \tag{2.1}$$

for any vector field X in TM.

On the other hand, by a theorem of Chen, Ludden and Montiel [5] it also satisfies the following

$$(\nabla_X A)Y = \eta(Y)\phi X + g(\phi X,Y)\xi. \tag{2.2}$$

Also (2.2) can be derived by the covariant differentiating of (2.1). From (2.1) and (2.2), together with (1.6), it follows that

$$(\nabla_X S)Y = 2n\{\eta(Y)\phi X + g(\phi X,Y)\xi\}. \tag{2.3}$$

So a horosphere in $H_n(C)$ satisfies the formula (0.1) for a non-zero constant $c = 2n$.

Next let us consider for the case where M is of type A_1 which is said to be geodesic hypersphere, that is, the tube of radius r over $H_k(C)$, $k=0$. Then its Weingarten map is given by

$$A=\begin{pmatrix} 2\coth r & & & 0 \\ & \coth r & & \\ & & \ddots & \\ 0 & & & \coth r \end{pmatrix}$$

Now let us denote a principal curvature $\coth r$ by t. Then the above Weingarten map A is represented by

$$AX = tX + \frac{1}{t}\eta(X)\xi,$$

and

$$(\nabla_X A)Y = \eta(Y)\phi X + g(\phi X, Y)\xi.$$

So it follows that

$$(\nabla_X A)AY = (t+\frac{1}{t})\eta(Y)\phi X + tg(\phi X, Y)\xi.$$

Substituting these formulas into (1.9), we get

(2.4)

$$\begin{aligned}(\nabla_X S)Y =& 3t\{g(\phi X, Y)\xi + \eta(Y)\phi X\} + (hI - A)\{\eta(Y)\phi X \\ &+ g(\phi X, Y)\xi\} - (t+\frac{1}{t})\eta(Y)\phi X - tg(\phi X, Y)\xi \\ =& 2nt\{g(\phi X, Y)\xi + \eta(Y)\phi X\}.\end{aligned}$$

Thus a geodesic hypersphere satisfies the formula (0.1) for $c = 2nt$.

From now on let us consider the converse problem. First of all in order to get a characterization of horospheres and geodesic hyperspheres we give the following lemma.

LEMMA 2.1. *Let M be a real hypersurface in $H_n(C)$. If the Ricci tensor S of M satisfies*

$$(0.1) \qquad (\nabla_X S)Y = c\{g(\phi X, Y)\xi + \eta(Y)\phi X\}$$

for a non-zero constant c, then the structure vector field ξ is principal.

Using the similar method given in [12], we can easily get the lemma above. So we omit this proof.

Now let us introduce the notion of *η-parallel Ricci tensor*, that is,

$$g((\nabla_X S)Y, Z) = 0$$

for any X, Y and Z in $\mathcal{D}$, where $\mathcal{D}$ denotes a distribution defined by $\mathcal{D}(x) = \{u \in T_x M : u \perp \xi(x)\}$ in the tangent space $T_x M$ of M at any point x in M. With this notion of η-parallel Ricci tensor Suh [15] classified real hypersurfaces in $H_n(C)$ as the following

THEOREM 2.2. *Let M be a real hypersurface in $H_n(C)$. Then the Ricci tensor of M is η-parallel and ξ is principal if and only if M is locally congruent to one of homogeneous real hypersurfaces of type A_0, A_1, A_2, and B.*

Now let us suppose that M is a real hypersurface in $H_n(C)$ satisfying the formula (0.1). Then Lemma 2.1 gives that its structure vector field ξ is principal.

On the other hand, from the formula (0.1) we have

$$g((\nabla_X S)Y, Z) = 0$$

for any vector fields X, Y and Z in $\mathcal{D}$. So it follows from these facts and Theorem 2.2 that M is locally congruent to one of homogeneous real hypersurfaces of type A_0, A_1, A_2 and B. Thus it remains only to show that the formula (0.1) does not hold for the case where M is of type A_2 and B.

If M is of type A_2, then its Weingarten map A is given by

$$A = \begin{pmatrix} 2\coth 2r & & & & & & 0 \\ & \coth r & & & & & \\ & & \ddots & & & & \\ & & & \coth r & & & \\ & & & & \tanh r & & \\ & & & & & \ddots & \\ 0 & & & & & & \tanh r \end{pmatrix}$$

where the principal curvatures $\coth r$ and $\tanh r$ have its multiplicities $2p$ and $2q$ respectively, and $p+q=n-1$. Now let us denote $\coth r$ by t. Then in this case $AX = tX$ implies $A\phi X = t\phi X$. So by a theorem of Chen, Montiel and Ludden [5] we get

$$(\nabla_X A)Y = \eta(Y)\phi X + g(\phi X, Y)\xi.$$

From this we have

$$(\nabla_X A)\phi X = \xi, \quad \textit{and} \quad (\nabla_X A)A\phi X = t\xi.$$

So it follows from this fact and (1.5) that

$$\begin{aligned} (\nabla_X S)\phi X &= 3g(\phi AX, \phi X)\xi + (hI - A)(\nabla_X A)\phi X - (\nabla_X A)A\phi X \\ &= 3t\xi + (hI - A)\xi - t\xi \qquad (2.5) \\ &= \{2(p+1)t + \frac{2q}{t}\}\xi, \end{aligned}$$

where we have used $(hI - A)\xi = (h - \alpha)\xi = (2pt + \frac{2q}{t})\xi$. Now let Y be the principal curvature vector orthogonal to ξ with the principal curvature $\frac{1}{t}$. Then by using the similar method as above we have

$$\begin{aligned} (\nabla_Y S)\phi Y &= 3g(\phi AY, \phi Y)\xi + (hI - A)(\nabla_Y A)\phi Y - (\nabla_Y A)A\phi Y \\ &= \frac{3}{t}\xi + (hI - A)\xi - \frac{1}{t}\xi \qquad (2.6) \\ &= \{2(q+1)\cdot\frac{1}{t} + 2pt\}\xi. \end{aligned}$$

Since M satisfies the formula (0.1), we know

$$(\nabla_X S)\phi X = c\xi = (\nabla_Y S)\phi Y.$$

So it follows from (2.5) and (2.6) that

$$\{2(p+1)t + \frac{2q}{t}\} = 2(q+1)\cdot\frac{1}{t} + 2pt.$$

Hence $t^2 = 1$. This contradicts the fact that $t = \coth r > 1$ or $t = \coth r < -1$. Thus this case can not occur.

Finally let us suppose that M is of type B. Then its Weingarten map A is given by

$$A = \begin{pmatrix} 2\tanh 2r & & & & & & 0 \\ & \coth r & & & & & \\ & & \ddots & & & & \\ & & & \coth r & & & \\ & & & & \tanh r & & \\ & & & & & \ddots & \\ 0 & & & & & & \tanh r \end{pmatrix}$$

where the principal curvatures $\coth r$ and $\tanh r$ have its multiplicities $n-1$ respectively. In this case its structure vector ξ is principal. That is $A\xi = \alpha\xi$. From this, taking differentiation, we get

$$A\phi A = -\phi + \frac{\alpha}{2}(A\phi + \phi A). \tag{2.7}$$

On the other hand, from (2.7) and the equation of Codazzi (1.7) we have

$$\begin{aligned}(\nabla_\xi A)X &= (\nabla_X A)\xi - \phi X = \alpha\phi AX - A\phi AX - \phi X \\ &= -\frac{\alpha}{2}(A\phi - \phi A)X.\end{aligned} \tag{2.8}$$

Since M satisfies (0.1), we have $\nabla_\xi S = 0$. So it follows from (1.8) that

$$\nabla_\xi S = h\nabla_\xi A - (\nabla_\xi A)A - A\nabla_\xi A = 0.$$

Substituting (2.8) into this equation, we get

$$h(A\phi - \phi A) - A^2\phi + \phi A^2 = 0. \tag{2.9}$$

That is the Ricci tensor S and the structure tensor ϕ commutes to each other. Now let us denote by $\alpha = 2\tanh 2r, \lambda = \tanh r$ and $\mu = \coth r$. Then (2.9) gives that

$$(\lambda - \mu)\{(\lambda + \mu) - h\} = 0. \tag{2.10}$$

Since the multiplicities of the principal curvatures λ and μ are equal to $n-1$ and $\lambda \neq \mu$, (2.10) gives

$$h = \alpha + (n-1)(\lambda + \mu) = \alpha + (n-1)h.$$

So $(n-2)h + \alpha = 0$, from which together with the fact that $\alpha = \frac{4\lambda}{(1+\lambda^2)}$ it follows that

$$4\lambda^2 + (n-2)(1+\lambda^2) = 0.$$

This contradicts. So this case can not occur. Summing up these facts, we have completed the proof of Theorem 1.

3. Proof of Theorem 2

Let M be a real hypersurface in a complex hyperbolic space $H_n(C)$. Motivated by Theorem 1 we will prove the main result in this section. From now on we will discuss our statement under the condition that the mean curvature h and the function $\eta(A\xi)$ which will be denoted by α are locally constant on M.

On the other hand, from (2.3) and (2.4) we know that the derivative of the Ricci tensor S of a horosphere and a geodesic hypersphere satisfies

$$(\nabla_X S)Y = \frac{n}{n-1}(h-\alpha)\{g(\phi X, Y)\xi + \eta(Y)\phi X\}. \tag{3.1}$$

Now let us define a new tensor field T on a real hypersurface M in $H_n(C)$ as follows

$$T(X,Y) = (\nabla_X S)Y - \frac{n}{n-1}(h-\alpha)\{g(\phi X, Y)\xi + \eta(Y)\phi X\}.$$

Thus by Theorem 1, $T = 0$ holds on M if and only if M is locally congruent to one of type A_0, A_1. For a case where $T = 0$ we here note that $h - \alpha \neq 0$. In fact, if $h = \alpha$, then the Ricci tensor S is parallel on M. But we know that there does not exist a real hypersurface in $H_n(C)$ with the parallel Ricci tensor ([7]).

Let $\{e_1, ..., e_{2n-1}\}$ be an orthonormal basis of $T_x(M)$ for any $x \in M$. Then the length of T can be calculated as follows

$$\begin{aligned} \|T\|^2 &= \Sigma_{i=1}^{2n-1} g(T(e_i, e_j), T(e_i, e_j)) \\ &= \|\nabla S\|^2 + \frac{4n^2}{n-1}(h-\alpha)^2 \\ &- \frac{2n}{n-1}(h-\alpha)\Sigma_{i,j} g((\nabla_{e_i} S)e_j, g(\phi e_i, e_j)\xi + \eta(e_j)\phi e_i), \end{aligned} \tag{3.2}$$

where we have used $\Sigma_{i,j}\|g(\phi e_i, e_j)\xi + \eta(e_j)\phi e_i\|^2 = 4(n-1)$.

On the other hand, by virtue of (1.9) we can calculate the following

$$\begin{aligned} &\Sigma_{i,j} g((\nabla_{e_i} S)e_j, g(\phi e_i, e_j)\xi + \eta(e_j)\phi e_i) = 2\Sigma_i g((\nabla_{e_i} S)\xi, \phi e_i) \\ &= 2\Sigma_i\{3g(\phi A e_i, \phi e_i) + g((hI - A)(\nabla_{e_i} A)\xi, \phi e_i) - g((\nabla_{e_i} A)A\xi, \phi e_i)\} \\ &= 6(h-\alpha) + 2Tr(\phi A \nabla_\xi A) + 4(n-1)h - 2(h-\alpha) - 4(n-1)\alpha \\ &= 4n(h-\alpha) + 2Tr(\phi A \nabla_\xi A), \end{aligned}$$

where we have used the constancy of the mean curvature h of M to the second equality and the equation of Codazzi (1.7) to the third equality. Then substituting this into (3.2), we get

$$\|T\|^2 = \|\nabla S\|^2 - \frac{4n}{n-1}(h-\alpha)\{n(h-\alpha) + Tr(\phi A \nabla_\xi A)\}.$$

Thus we have proved Theorem 2 in §0.

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