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BIFURCATIONS IN A DISCRETE NONLINEAR DIFFUSION EQUATION

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ABSTRACT. We consider an infinite dimensional dynamical system what is called Lattice Dynamical System given by a discrete nonlinear diffusion equation. By assuming the nonlinearity to be a general nonlinear function with mild restrictions, we show that as the diffusion parameter changes the stationery states of the given system undergoes bifurcations from the zero state to a bounded invariant set or a 3- or 4-periodic state in the global phase space of the given system according to the values of the coefficient of the linear part of the given nonlinearity.

1. Introduction

Over the past ten years, a new class of infinite dimensional dynamical systems, so called Lattice Dynamical Systems (LDS), have been studied by many researchers. These systems proved to be a very useful tool for the investigation of behavior of physical systems with particle-like localized unbounded media. They are also effectively used in computer simulations of discretized partial differential equations [1, 2, 3].

Now suppose that at each site j of a d-dimensional lattice $\mathbf{Z}^d$, we have a finite dimensional local dynamical system which is defined by some map $f_j : M_j \longrightarrow M_j$, where M_j is a local phase space at the site j. For simplicity, we consider an infinite chain ($d = 1$) and $M_j = \mathbf{E}^p \, \forall j \in \mathbf{Z}$, where $\mathbf{E}^p$ is a p-dimensional Euclidean space with ordinary inner product $(\cdot,\cdot)$ and the norm $|\cdot| = \sqrt{(\cdot,\cdot)}$. Then we have an infinite dimensional dynamical system with the phase space $M = \prod_{j\in\mathbf{Z}} M_j$ and a

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point $u \in M$ can be thought of as a biinfinite sequence $u = \{u_j\}$, where $u_j \in M_j,\ j \in \mathbf{Z}$. To make the linear space M (with componentwise addition and scalar multiplication) to be a Hilbert space, we equip M with the inner product defined by

$$\langle u, v\rangle_q = \sum_{j\in\mathbf{Z}} \frac{(u_j, v_j)}{q^{|j|}} \quad \forall u, v \in M,$$

where $q > 1$ is some fixed number depending on the particular problem.

Define $\|\cdot\|_q = \sqrt{\langle\cdot,\cdot\rangle_q}$ and $B_q = \{u \in M \mid \|u\|_q < \infty\}$. Then it can be easily shown that B_q is a Hilbert space [1].

DEFINITION 1. Define the evolution operator $\Phi : B_q \longrightarrow B_q$ by

$$(\Phi u)_j = H(\{u_j\}^s), \tag{1}$$

where $\{u_j\}^s = \{u_i \mid |i-j| \leq s, j \in \mathbf{Z}, s \geq 1\,\text{integer}\}$ and $H : (\mathbf{E}^p)^{2s+1} \longrightarrow \mathbf{E}^p$ is a differentiable map of class C^2 such that

$$\left|\frac{\partial H}{\partial u_i}\right| \leq A, \quad \left|\frac{\partial^2 H}{\partial u_i \partial u_j}\right| \leq A, \tag{2}$$

for any collection $\{u_j\}^s$ and some constant A.

Then it is easy to verify that under the condition (2), $\Phi(B_q) \subset B_q$ and Φ is Lipschitz continuous with the constant $L = C(2s+1)^{\frac{3}{2}} q^{\frac{s}{2}}$ [1].

DEFINITION 2. Given a state $u(n) = \{u_j(n)\}_{j=-\infty}^{\infty} \in B_q$ at the moment n, we can obtain via (1) the next state $u(n+1)$, that is,

$$\begin{aligned} u(n+1) &= \Phi(u(n)) \quad \text{or} \\ u_j(n+1) &= (\Phi(u(n)))_j = H(\{u_j(n)\}^s). \end{aligned} \tag{3}$$

The dynamical system $(\Phi^n, B_q)_{n\in\mathbf{Z}^+}$ is called a *lattice dynamical system* (LDS).

DEFINITION 3. A state (or solution) $u_j(n)$ for (3) is *spatially homogeneous* if $u_j(n) = \xi(n)\,\forall j \in \mathbf{Z}$ and is *stationary* if $u_j(n) = \psi_j\,\forall n \in \mathbf{Z}^+$. If $\Phi^m u = u$ for some $m \in \mathbf{Z}^+$, then u is *time m-periodic* and $u_{j+k} = u_j \forall j \in \mathbf{Z}$ for some $k \in \mathbf{Z}^+$, then u is *space k-periodic.* If u is both time m-periodic and space k-periodic, then we shall briefly call such u an *(m,k)-periodic* solution.

For instance, the spatially homogeneous stationary solutions are $(1,1)$-solutions.

DEFINITION 4. The translational group $\{S^j\}_{j\in\mathbf{Z}}$ acts on B_q by

$$(4) \qquad (S^{j_0}u)_j = u_{j+j_0},$$

where $S : B_q \longrightarrow B_q$ is a shift operator defined by

$$(Su)_j = u_{j+1}$$

and S^j is the nth iteration of S. The dynamical system $(\{S^j\}, B_q)_{j\in\mathbf{Z}}$ is called a *translational dynamical system* (TDS).

Obviously, the TDS $\{S^j\}_{j\in\mathbf{Z}}$ is generated by the shift map $S^1 = S$.

2. Discrete diffusion equations with general nonlinearity

Consider a discrete version of one-dimensional nonlinear diffusion equation of the form

$$(5) \quad u_j(n+1) = u_j(n) + f(u_j(n)) + ((1+\varepsilon)u_{j-1}(n) - 2u_j(n) + u_{j+1}(n)),$$

where ε is a sufficiently small real parameter and it represents a symmetric or asymmetric diffusion coupling according to $\varepsilon = 0$ or $\varepsilon \neq 0$ respectively, and f is a nonlinear function of class C^∞ in the form

$$(6) \quad \begin{aligned} &f(u) = au + \mathcal{O}(|u|^2), \quad 0 < a < 4 \quad \text{for} \quad |u| \leq R, R \gg 1, \quad \text{and} \\ &|f'(u)| \leq A, \quad |f''(u)| \leq A \quad \forall u \in \mathbf{R} \quad \text{and for some constant } A. \end{aligned}$$

In this paper, we will restrict our attention to stationary solutions of (5) and investigate bifurcation phenomena of them as ε varies. Setting

$u_j(n) = \psi_j \, \forall j \in \mathbf{Z}, n \in \mathbf{Z}^+$, (5) becomes a second order difference equation

$$f(\psi_j) + (1 + \varepsilon)\psi_{j-1} - 2\psi_j + \psi_{j+1} = 0. \tag{7}$$

and putting again

$$x_j = \psi_{j-1}, y_j = \psi_j \tag{8}$$

we obtain a 2D discrete dynamical system

$$\begin{aligned} x_{j+1} &= y_j \\ y_{j+1} &= 2y_j - (1+\varepsilon)x_j - f(y_j) \end{aligned} \tag{9}$$

This system is generated by the Hénon-type map

$$F_\varepsilon : (x, y) \longrightarrow (y, 2y - (1+\varepsilon)x - f(y)) \tag{10}$$

and, in fact, a TDS on the set of stationary states of (5).

Note that bounded orbits of (9) $(\cdots, (x_j, y_j), (x_{j+1}, y_{j+1}), \cdots)$ are bounded solutions of (7) $\psi = (\cdots, \psi_j = y_j, \psi_{j+1} = y_{j+1}, \cdots)$ which are in turn bounded stationary states $\psi \in B_q$ of (5). Now, we investigate the bounded orbits of (9). Let

$$K = \{(x, y) \in \mathbf{R}^2 \,|\, |F_\varepsilon^j(x, y)| < \infty \, \forall j \in \mathbf{Z}\}.$$

Define a map $h : K \longrightarrow B_q$ by

$$(h(x, y))_j = \pi_2 \circ F_\varepsilon^j(x, y) \quad \forall (x, y) \in K, j \in \mathbf{Z},$$

where π_2 is a projection onto the y-axis in the (x, y)-plane. Then, $h : K \longrightarrow h(K) \subset B_q$ is a homomorphism and $h \circ F_\varepsilon = S \circ h$, i.e., $F_\varepsilon|_K$ and $S|_{h(K)}$ are topologically conjugate.

3. Bifurcation analysis

Consider again the Hénon-type map (10)

$$F_\varepsilon : (x,y) \longrightarrow (y, 2y - (1+\varepsilon)x - f(y)),$$

where f satisfies the conditions (6). Notice that $(0,0)$ is a fixed point of $F_\varepsilon \, \forall \varepsilon \in \mathbf{R}$ and via (8) it corresponds to a spatially homogeneous stationary state $u_j(n) = 0 \, \forall j \in \mathbf{Z},\ n \in \mathbf{Z}^+$ in (5). When $\varepsilon = 0$, the linear part $DF_0(0,0)$ of the map F_0 at $(0,0)$ has complex conjugate eigenvalues $\lambda_0, \bar{\lambda}_0$ with $|\lambda_0| = 1$ and

$$\lambda_0 = \frac{1}{2}\Big[(2-a) + i\sqrt{a(4-a)}\Big]. \tag{11}$$

Here we assume that $0 < a < 4$ and $a \neq 1,2,3$ so that $\lambda_0^n \neq 1$ for $n = 1,2,3,4$.

When $\varepsilon \neq 0$, let $A_\varepsilon = DF_\varepsilon(0,0)$. Then A_ε has also complex conjugate eigenvalues $\lambda(\varepsilon),\ \bar{\lambda}(\varepsilon)$ with $\lambda(0) = \lambda_0$ if $|\varepsilon|$ is sufficiently small so that $|\varepsilon| < a(4-a)/4$. Moreover, we note that $|\lambda(\varepsilon)| = \sqrt{1+\varepsilon}$ and so $\frac{d}{d\varepsilon}|\lambda(\varepsilon)|\big|_{\varepsilon=0} = \frac{1}{2} > 0$. In other words, the map F_ε satisfies the Hopf condition at weak resonance.

Since F_ε is at least of class C^2 near $(0,0)$, λ_ε is at least of class C^1 and we can write

$$\lambda(\varepsilon) = \lambda_0(1 + \lambda_1\varepsilon + \mathcal{O}(|\varepsilon|^2)). \tag{12}$$

Let

$$\lambda_0 = e^{i2\pi\theta_0},\ \lambda_1 = Re\lambda_1 + i2\pi\theta_1,\ \lambda(\varepsilon) = |\lambda(\varepsilon)|e^{i2\pi\theta(\varepsilon)}.$$

Then from (12), we can write

$$\begin{aligned} |\lambda(\varepsilon)| &= 1 + \varepsilon Re\lambda_1 + \mathcal{O}(|\varepsilon|^2), \\ \theta(\varepsilon) &= \theta_0 + \varepsilon\theta_1 + \mathcal{O}(|\varepsilon|^2), \end{aligned} \tag{13}$$

where $Re\lambda_1 > 0$ since $\frac{d}{d\varepsilon}|\lambda_\varepsilon|\big|_{\varepsilon=0} > 0$. With slight abuse of notation, let us write

$$F_\varepsilon(x) = A_\varepsilon x + \mathcal{O}(|x|^2), \quad \text{where} \quad x = (x_1, x_2) \in \mathbf{R}^2. \tag{14}$$

Note that the higher order term is in general form because of the arbitrariness of the nonlinearity f in (5). Since A_ε has complex conjugate eigenvalues $\lambda(\varepsilon), \bar{\lambda}(\varepsilon)$ with $\lambda(\varepsilon) = |\lambda(\varepsilon)|e^{i2\pi\theta(\varepsilon)}$, it may be put in the Jordan form by a linear transformation and henceforth we may assume that (14) has been written in this form with

$$A_\varepsilon = |\lambda(\varepsilon)| \cdot \begin{bmatrix} \cos 2\pi\theta(\varepsilon) & -\sin 2\pi\theta(\varepsilon) \\ \sin 2\pi\theta(\varepsilon) & \cos 2\pi\theta(\varepsilon) \end{bmatrix}. \tag{15}$$

Now we may identify $\mathbf{R}^2$and $\mathbf{C}$ by setting $z = x_1 + ix_2$, and considering $z, \bar{z}$ as independent variables.

Then (14) can be rewritten in the following complex form, again denated as F_ε,

$$F_\varepsilon(z) = \lambda(\varepsilon)z + R(z, \bar{z}, \varepsilon), \tag{16}$$

where $\lambda(\varepsilon)$ satisfies (13) and $R(z, \bar{z}, \varepsilon) = \mathcal{O}(|z|^2)$.

Now, according to the theory of normal form in the case of weak resonance ($\lambda_0^n \neq 1, n = 1, 2, 3, 4$), [4], there exists a C^∞ ε-dependent change of coordinates such that in the new coordinates F_ε has the form, again denoted as F_ε,

$$F_\varepsilon(z) = \lambda(\varepsilon)z + \alpha(\varepsilon)z^2\bar{z} + \beta(\varepsilon)\bar{z}^4 + R_5(z, \bar{z}, \varepsilon), \tag{17}$$

where $R_5(z, \bar{z}, \varepsilon) = \mathcal{O}(|z|^5)$ and one can make $\beta(\varepsilon) = 0$ if $\lambda_0^5 \neq 1$.

We first consider the case of weak resonance. Writing again F_ε in polar coordinates $z = re^{i2\pi\phi}$, $F_\varepsilon(z) = Re^{i2\pi\Phi}$ and after some calculations we obtain

(18)
$$\begin{cases} R = (1 + \varepsilon Re\lambda_1)r - \alpha r^3 + Re(\beta_0\bar{\lambda}_0 e^{-i10\pi\phi})r^4 + \mathcal{O}(|\varepsilon|^2 r + |\varepsilon| r^3 + r^5), \\ \Phi = \phi + \theta_0 + \varepsilon\theta_1 + \omega_1 r^2 + \mathcal{O}(|\varepsilon|^2 + |\varepsilon| r^2 + r^3) \quad (\text{mod } 1), \end{cases}$$

where $\alpha_0 = \alpha(0)$, $\beta_0 = \beta(0)$ and $\alpha = -Re(\alpha_0\bar{\lambda}_0)$, $\omega_1 = \frac{1}{2\pi}Im(\alpha_0\bar{\lambda}_0)$. We assume that $\alpha \neq 0$. When $\varepsilon = 0$, the first equation of (18) becomes

$$R = r(1 - \alpha r^2) + \mathcal{O}(r^4) \tag{19}$$

and by simple graphical analysis we know that the fixed point $r = 0$ is asymptotically stable if $\alpha > 0$ and is unstable if $\alpha < 0$. When $\varepsilon \neq 0$, from (18), we have a fixed point $r = 0$ and invariant circles $\varepsilon = \frac{\alpha}{Re\lambda_1} r^2 + \mathcal{O}(r^3)$. When $\alpha > 0$, the fixed point $r = 0$ is asymptotically stable for $\varepsilon \leqslant 0$ and becomes unstable for $\varepsilon > 0$, when a stable attracting invariant circle bifurcates from the origin $r = 0$. When $\alpha < 0$, the fixed point $r = 0$ is unstable for $\varepsilon \geq 0$ and becomes stable for $\varepsilon < 0$, when an unstable (repelling) invariant circle bifurcates from $r = 0$.

Returning to the map (9) and our original equation (5), the fixed point $r = 0$ corresponds to the spatially homogeneous stationary state $\psi_j = 0 \, \forall j \in \mathbf{Z}$ and the invariant circle corresponds to the bounded invariant set $S = \{\psi \in B_q | \, \|\psi\|_\infty \leq r\}$ in B_q. Summarizing the above analysis, we obtain the following results.

THEOREM 1. *Suppose that the nonlinearity f of (5) is of class C^∞ and satisfies the conditions (6) and assume that $0 < a < 4$ and $a \neq 1, 2, 3$. Suppose also that the associated Hénon-type map (10) has been put in a complex normal form*

$$F_\varepsilon(z) = \lambda(\varepsilon) + \alpha(\varepsilon) z^2 \bar{z} + \mathcal{O}(|z|^4),$$

where $\lambda(\varepsilon) = \lambda_0(1 + \lambda_1 \varepsilon + \mathcal{O}(|\varepsilon|^2))$ and $\lambda_0 = \lambda(0)$. Assume that $\alpha = -Re(\alpha_0 \bar{\lambda}_0) \neq 0$, where $\alpha_0 = \alpha(0)$. Then, when $\alpha > 0$, the zero stationary state $u = 0$ of (5) is asymptotically stable for $\varepsilon \leqslant 0$, and becomes unstable for $\varepsilon > 0$ and an attracting invariant bounded set in B_q of the form

$$\{u \in B_q | \, \|u\|_\infty \leq r\}, \; r = \sqrt{\frac{\varepsilon Re\lambda_1}{\alpha}} + o(|\varepsilon|^{\frac{1}{2}})$$

bifurcates from the zero state $u = 0$.

When $\alpha < 0$, the zero stationary $u = 0$ of (5) is unstable for $\varepsilon \geqslant 0$, and become stable for $\varepsilon < 0$ and a repelling invariant bounded set of the above form bifurcates from the zero state $u = 0$.

Next, we consider the case of strong resonance. In this case, the normal form of the associated Hénon-type map takes the form

$$F_\varepsilon(z) = \lambda(\varepsilon) z + c_{02}(\varepsilon) \bar{z}^2 + c_{21}(\varepsilon) z^2 \bar{z} + \mathcal{O}(|z|^4) \tag{20}$$

when $\lambda_0^3 = 1$ and

$$F_\varepsilon(z) = \lambda(\varepsilon)z + d_{21}(\varepsilon)z^2\bar{z} + d_{03}(\varepsilon)\bar{z}^3 + \mathcal{O}(|z|^5) \tag{21}$$

when $\lambda_0^4 = 1$, respectively [4]. To examine the periodic orbits of $F_\varepsilon(z)$, write the equation $F_\varepsilon^n(z) = z$ in the equivalent system

$$F_\varepsilon(x_i) = x_{i+1},\ i = 1, \cdots, n-1,\ F_\varepsilon(x_n) = x_1, \tag{22}$$

where $\{x_i\}_{i=1}^n \subset \mathbf{C}$, is a n-cycle of F_ε. Rewriting (22) in the vector-matrix form, we have

$$Sx = \mathcal{F}_\varepsilon(x), \tag{23}$$

where $x = (x_1, x_2, \cdots, x_n) \in \mathbf{C}^n$, $\mathcal{F}_\varepsilon(x) = (F_\varepsilon(x_1), \cdots, F_\varepsilon(x_n)) \in \mathbf{C}^n$, and

$$S = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & 1 \\ 1 & 0 & 0 & \dots & 0 \end{bmatrix}$$

and diagonalizing S by a linear change of coordinates $y = Px$, (23) can be written as

$$\Phi(y, \varepsilon) = P\mathcal{F}_\varepsilon(P^{-1}y) - \Lambda y = 0, \tag{24}$$

where

$$\Lambda = PSP^{-1} = diag(1, \bar{\lambda_0}, \bar{\lambda_0}^2, \dots, \bar{\lambda_0}^{n-1}), \quad \Phi : \mathbf{C}^n \times \mathbf{R} \longrightarrow \mathbf{C}^n.$$

Since the linear part $L \equiv Dy\Phi(0,0)$ has the kernel which is a $1D$ subspace of $\mathbf{C}^n$, we can apply the Liapunov-Schmidt method [5] to obtain a bifurcation function

$$g(y_n, \varepsilon) \equiv \langle \Phi(y, \varepsilon), v_n \rangle = 0, \tag{25}$$

where $y = y_n v_n + \theta(y_n, \varepsilon)$, $v_n = (0, \cdots, 0, 1) \in KerL$, $\theta(y_n, \varepsilon) \in (KerL)^\perp$ and we know that $\theta(y_n, \varepsilon) = \mathcal{O}(|\varepsilon| \cdot |y_n| + |y_n|^2)$ by implicit differentiation. Also letting $z = \frac{1}{n}y_n$, one can easily show that equation (25) is equivalent to the equation

$$\lambda_0 z = F_\varepsilon(z) = \lambda(\varepsilon)z + R(z, \bar{z}, \varepsilon), \tag{26}$$

where $R(z, \bar{z}, \varepsilon)$ is in normal form and satisfies the relation $R(\lambda_0 z, \overline{\lambda_0 z_0}, \varepsilon) = \lambda_0 R(z, \bar{z}, \varepsilon)$. Note that the solution z in (26) is not the fixed point z in (22). The fixed point of F_ε^n is x_1 which is given by

$$\begin{aligned} x_1 = (p^{-1}y)_1 &= \frac{1}{n}(y_n + \sum_{i=1}^{n-1} \theta_i(y_n, \varepsilon)) \\ &= z + \frac{1}{n}\sum_{i=1}^{n-1} \theta_i(nz, \varepsilon) = z + \mathcal{O}(|\varepsilon| \cdot |z| + |z|^2), \end{aligned} \tag{27}$$

where (z, ε) is a solution of (26). Also note that if (z, ε) is a solution of (26), then $(\lambda_0^k z, \varepsilon)(k = 0, 1, \cdots, n-1)$ is also a solution of (26) which gives n fixed points $x_1, x_2, \cdots, x_n$ of (22), where

$$x_k = \lambda_0^{k-1} z + \mathcal{O}(|\varepsilon| \cdot |z| + |z|^2). \tag{28}$$

Now our problem of finding fixed points of (22) has been reduced to solving the equation (26), where $F_\varepsilon(z)$ is in normal form.

Consider first the case $\lambda_0^3 = 1$. The normal form of $F_\varepsilon(z)$ in this case is

$$F_\varepsilon(z) = \lambda(\varepsilon)z + c_{02}(\varepsilon)\bar{z}^2 + c_{21}(\varepsilon)z^2\bar{z} + \mathcal{O}(|z|^4).$$

Writing $\lambda(\varepsilon) = \lambda_0(1 + \varepsilon\lambda_1 + \mathcal{O}(|\varepsilon|^2))$, the equation (26) becomes

$$\varepsilon\lambda_1 z + \bar{\lambda}_0 \beta \bar{z}^2 + \mathcal{O}(|\varepsilon|^2|z| + |\varepsilon||z|^2 + |z|^3) = 0, \tag{29}$$

where $\beta = c_{02}(0)$ and we assume that $\beta \neq 0$.

Letting $z = re^{2\pi i\phi}$ in (29), we obtain $r = 0$ and

$$\varepsilon\lambda_1 + \bar{\lambda}_0 \beta e^{-6\pi i\phi} r + \frac{1}{n} g(\varepsilon, r, \phi) = 0, \tag{30}$$

where $g \in C^\infty$, $g(\varepsilon, r, \phi + \frac{1}{3}) = g(\varepsilon, r, \phi) = \mathcal{O}(r(|\varepsilon| + r)^2)$. Set

$$
(31) \qquad \begin{cases} |\varepsilon| = |\dfrac{\beta}{\lambda_1}| r(1+\varepsilon_1), \\ \phi = \phi_0 + \phi_1, \\ \phi_0 = \begin{cases} \frac{1}{6\pi} \arg\left(-\frac{\bar{\lambda_0}\beta}{\lambda_1}\right) \ (\bmod\ \frac{1}{3}), & \text{for } \varepsilon > 0 \\ -\frac{1}{6} + \frac{1}{6\pi} \arg\left(-\frac{\bar{\lambda_0}\beta}{\lambda_1}\right) (\bmod\ \frac{1}{3}), & \text{for } \varepsilon < 0, \end{cases} \end{cases}
$$

where $\varepsilon_1 = \varepsilon_1(r)$, $\phi_1 = \phi_1(r)$ are to be determined. Now applying the implicit function theorem, we can easily show that $\varepsilon_1(r) = \mathcal{O}(r)$ and $\phi_1(r) = \mathcal{O}(r)$. Hence we obtain the 3-cycle given by

$$
(32) \qquad \begin{aligned} x_1 &= z(r) + \frac{1}{3}\sum_{i=1}^{2} \theta_i(3z(r), \varepsilon(r)) = z(r) + \mathcal{O}(r^2), \\ x_2 &= \lambda_0 z(r) + \mathcal{O}(r^2), \\ x_3 &= \bar{\lambda_0} z(r) + \mathcal{O}(r^2), \end{aligned}
$$

where $\lambda_0 = e^{2\pi i/3}$ and r depends on ε by (31).

Note that from (31), when $\varepsilon < 0$, we have the 3-cycle with the phase delayed by $\frac{\pi}{3}$ and so the orientation is reversed from the case when $\varepsilon > 0$. Returning to the map (9) and LDS (5), we can state the following theorem.

THEOREM 2. *Suppose that the nonlinearity f of (5) satisfies the conditions as in (6). Assume that $a = 3$ which implies $\lambda_0^3 = 1$. Suppose also that the associated Hénon-type map (10) has been put in a complex normal form*

$$
F_\varepsilon(z) = \lambda(\varepsilon) + c_{02}(\varepsilon)\bar{z}^2 + c_{21} z^2 \bar{z} + \mathcal{O}(|z|^4),
$$

where $\beta = c_{02}(0)$ is also assumed to be $\beta \neq 0$. Then, as ε passes through $\varepsilon = 0$, the zero state $u = 0$ of the LDS (5) bifurcates to a 1-parameter family of 3-periodic orbits on both sides of $\varepsilon = 0$, where the 3-periodic orbit $\{\psi_i \mid \psi_{i+3} = \psi_i, i \in \mathbf{Z}\}$ is given by

$$
\psi_i = \lambda_0^{i-1} r \sin 2\pi\phi + \mathcal{O}(r^2), \ i = 1, 2, 3
$$

with r and ϕ depends on ε by (31).

In the case of $\lambda_0^4 = 1$, we follow the same procedure as above and obtain the following results.

THEOREM 3. *Suppose that the nonlinearity f of (5) satisfies the conditions as in (6) and that $a = 2$ so that $\lambda_0^4 = 1$. Suppose also that the associated Hénon-type map (10) has been put in a complex normal form*

$$F_\varepsilon(z) = \lambda(\varepsilon)z + d_{21}(\varepsilon)z^2\bar{z} + d_{03}(\varepsilon)\bar{z}^3 + \mathcal{O}(|z|^5),$$

where $z = re^{2\pi i\phi}$ and we set $a_1 = \lambda_0 d_{21}(0)$, $a_2 = \bar{\lambda}_0 d_{03}(0)$ and assume that $Im\left(\frac{a_1}{\lambda_1}\right) < \left|\frac{a_2}{\lambda_1}\right|$ (otherwise there does not exist any 4-periodic orbits bifurcating from $u = 0$). Then, as ε passes through $\varepsilon = 0$, the zero state $u = 0$ of the LDS (5) bifurcates to a pair of one-parameter families of 4-periodic orbits $\{\psi_i^{(1)}\}, \{\psi_i^{(2)}\}$ on the same side of $\varepsilon > 0$ if $|a_1| > |a_2|$ and $Re\left(\frac{a_1}{\lambda_1}\right) < 0$ and on the same side of $\varepsilon < 0$ if $|a_1| > |a_2|$ and $Re\left(\frac{a_1}{\lambda_1}\right) > 0$ and on the opposite side of $\varepsilon = 0$ if $|a_1| < |a_2|$. Furthermore, the 4-periodic orbits are given by

$$\psi_i^{(j)} = \lambda_0^{i-1} r \sin 2\pi\phi^{(j)}, \quad i = 1,2,3,4, \quad j = 1,2,$$

where r and ϕ depend on ε by the relation

$$\varepsilon^{(j)} = \varepsilon_0^{(j)} r^2 + \mathcal{O}(r^4), \quad j = 1,2$$

$$\phi^{(j)} = \phi_0^{(j)} + \mathcal{O}(r^2), \quad j = 1,2 \quad \text{and}$$

$$\varepsilon_0^{(j)} = -Re\left(\frac{a_1}{\lambda_1}\right) + (-1)^{j-1}\left[\left|\frac{a_2}{\lambda_1}\right|^2 - \left\{Im\left(\frac{a_1}{\lambda_1}\right)\right\}^2\right]^{\frac{1}{2}}, \quad j = 1,2,$$

$$\phi_0^{(j)} = -\frac{1}{8\pi}\arg\left[-\frac{\varepsilon_0^{(j)}\lambda_1 + a_1}{a_2}\right] + \mathcal{O}(r^2) \quad (\text{mod}\,\frac{1}{4}), \quad j = 1,2.$$

References

[1] V. S. Afraimovich and L. A. Bunimovich, *Simplest structures in coupled map lattices and their stability*, preprint in GIT (1992).
[2] K. Kaneko, *Theory and applications of coupled map lattices*, John-Wiley and Sons, 1993.
[3] L. A. Bunimovich et al., *Trivial Maps*, Chaos **2** (1992).
[4] G. Ioose, *Bifurcation of maps and applications*, North-Holland, 1979.
[5] S. N. Chow and J. K. Hale, *Methods of Bifurcation Theory*, Springer-Verlag, 1982.

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