Bull. Korean Math. Soc. **35** (1998), No. 4, pp. 653–658

THE STABILITY OF THE EQUATION $f(x+p)=kf(x)$

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ABSTRACT. In this paper, we investigate the Hyers-Ulam stability of the (p,k)-MP functional equation.

1. Introduction

The stability problem of functional equations has been originally raised by S. M. Ulam [5]. In 1940, he had raised the following question: Under what conditions does there exist an additive mapping near an approximately additive mapping? In 1941, this problem was solved by D. H. Hyers [1] for the first time. This problem has been further generalized and solved by Th. M. Rassias [4]. Thereafter, the stability problem of functional equations has been extended in various directions and studied by several mathematicians [2].

In this paper, the Hyers-Ulam stability of the (p,k)-MP functional equation (1) is investigated. Furthermore, a modified Hyers-Ulam-Rassias stability of the functional equation (9) shall also be investigated.

2. Hyers-Ulam stability of the (p,k)-MP functional equation

The following functional equation

$$f(x+p)=kf(x) \tag{1}$$

is called the (p,k)-MP functional equation. Throughout this section, let $\delta>0,\ k>0$, and $p\neq 0$ be fixed.

Received March 31, 1998.
1991 Mathematics Subject Classification: Primary 39B72, 39B22.
Key words and phrases: (p,k)-MP functional equation, stability.

THEOREM 1. *Let $k \neq 1$. If a mapping $f : \mathbb{R} \to \mathbb{R}$ satisfies the following inequality*

$$|f(x+p) - kf(x)| \leq \delta \tag{2}$$

for all $x \in \mathbb{R}$, then there exists a unique solution $F : \mathbb{R} \to \mathbb{R}$ of the (p,k)-MP functional equation (1) *with*

$$|F(x) - f(x)| \leq |k-1|^{-1}\delta \tag{3}$$

for all $x \in \mathbb{R}$.

Proof. (I) The case of $1 < k$: For any $x \in \mathbb{R}$ and for every non-negative integer n we define

$$P_n(x) = k^{-n} f(x + pn).$$

Then $P_0(x) = f(x)$. By (2) we have

$$|P_{n+1}(x) - P_n(x)| \leq k^{-(n+1)}\delta,$$

whence

$$|P_n(x) - f(x)| \leq (k-1)^{-1}\delta. \tag{4}$$

Let $m \leq n$. Then

$$|P_n(x) - P_m(x)| \leq k^{-m}(k-1)^{-1}\delta,$$

whence $|P_n(x) - P_m(x)| \to 0$ as $m, n \to \infty$ since $k > 1$. This fact implies that $\{P_n(x)\}$ is a Cauchy sequence for all $x \in \mathbb{R}$ and hence we can define a mapping $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = \lim_{n\to\infty} P_n(x)$$

for all $x \in \mathbb{R}$. Then by (4), F satisfies the inequality (3) and

$$F(x+p) = \lim_{n\to\infty} P_{n-1}(x+p) = k \lim_{n\to\infty} P_n(x) = kF(x) \tag{5}$$

for all $x \in \mathbb{R}$. Thus $F(x+pn) = k^n F(x)$, so $F(x) = k^{-n}F(x+pn)$. Now, let $G : \mathbb{R} \to \mathbb{R}$ be another mapping which satisfies (5) as well as (3) for all $x \in \mathbb{R}$. It follows from (5) and (3) that

$$\begin{aligned}|F(x) - G(x)| &= |k^{-n}F(x+pn) - k^{-n}G(x+pn)| \\ &\le k^{-n}(|F(x+pn) - f(x+pn)| \\ &\qquad + |G(x+pn) - f(x+pn)|) \\ &\le k^{-n}((k-1)^{-1}\delta + (k-1)^{-1}\delta)\end{aligned}$$

for all $x \in \mathbb{R}$ and all positive integers n. Thus $F(x) = G(x)$.

(II) The case of $0 < k < 1$: An equivalent formula of inequality (2) is

$$|f(x-p) - k^{-1}f(x)| \le k^{-1}\delta. \tag{6}$$

By the proof of the case (I), there exists unique $F : \mathbb{R} \to \mathbb{R}$ such that

$$F(x-p) = k^{-1}F(x) \tag{7}$$

and

$$|F(x) - f(x)| \le \frac{1}{k^{-1}-1}k^{-1}\delta = \frac{1}{1-k}\delta. \tag{8}$$

An equivalent form of (7) is

$$F(x+p) = kF(x).$$

The proof is complete. □

The following example shows that the above theorem is false when $k = 1$.

EXAMPLE 2. Let $f(x) = x$ and $p = 1$, $\delta = 1$, $k = 1$. Then $|f(x+1) - f(x)| = 1 = \delta$. Assume that $F(x)$ is a solution of the $(1,1)$-MP functional equation (1). If $F(0) = c$, then $F(n) = c$ for all $n \in N$. Thus $|F(n) - f(n)| = |c - n| \to \infty$ as $n \to \infty$.

3. A modified Hyers-Ulam-Rassias stability of the (p,k)-MP functional equation

Let $\delta,\ \epsilon,\ p>0$ be given and define

$$\alpha(x)=\prod_{i=0}^{\infty}[1-\delta(x+pi)^{-(1+\epsilon)}],\ \beta(x)=\prod_{i=0}^{\infty}[1+\delta(x+pi)^{-(1+\epsilon)}]$$

for any $x>\delta^{1/(1+\epsilon)}$. Let $n_0\geq 0$ be any integer. By using an idea from [3], we can prove the following theorem:

THEOREM 3. *Let $0<k$. If a mapping $f:(0,\infty)\to(0,\infty)$ satisfies the inequality*

$$\left|\frac{f(x+p)}{kf(x)}-1\right|\leq\frac{\delta}{x^{1+\epsilon}} \tag{9}$$

for all $x>n_0$, then there exists a unique solution $F:(0,\infty)\to[0,\infty)$ of the (p,k)-MP functional equation (1) with

$$\alpha(x)\leq F(x)/f(x)\leq\beta(x) \tag{10}$$

for all $x>\max\{n_0,\ \delta^{1/(1+\epsilon)}\}$.

Proof. Let $P_n(x)$ be defined as in the proof of Theorem 1. For any $x>0$ and for all positive integers $m,\ n$ with $n>m$, it holds

$$\frac{P_n(x)}{P_m(x)}=\frac{f(x+p(m+1))}{kf(x+pm)}\frac{f(x+p(m+2))}{kf(x+p(m+1))}\cdots\frac{f(x+pn)}{kf(x+p(n-1))}.$$

If $m(>n_0)$ is so large that $1-\delta(x+pm)^{-(1+\epsilon)}>0$, we then obtain

$$\prod_{i=m}^{n-1}[1-\delta(x+pi)^{-(1+\epsilon)}]\leq P_n(x)/P_m(x)\leq\prod_{i=m}^{n-1}[1+\delta(x+pi)^{-(1+\epsilon)}]$$

or

$$\sum_{i=m}^{n-1}\ln\,[1-\delta(x+pi)^{-(1+\epsilon)}]\leq\ln\,P_n(x)-\ln\,P_m(x)$$
$$\leq\sum_{i=m}^{n-1}\ln\,[1+\delta(x+pi)^{-(1+\epsilon)}].$$

Since

$$\lim_{m\to\infty} \sum_{i=m}^{\infty} |\ln [1-\delta(x+pi)^{-(1+\epsilon)}]| = \lim_{m\to\infty} \sum_{i=m}^{\infty} \ln [1+\delta(x+pi)^{-(1+\epsilon)}] = 0,$$

we conclude that $\{\ln P_n(x)\}$ is a Cauchy sequence for all $x > 0$. Hence, we can define

$$L(x) = \lim_{n\to\infty} \ln P_n(x)$$

and

$$F(x) = e^{L(x)}$$

for all $x > 0$. So, $F(x) = \lim_{n\to\infty} P_n(x)$ and

$$F(x+p) = \lim_{n\to\infty} P_n(x+p) = \lim_{n\to\infty} kP_{n+1}(x) = kF(x)$$

for all $x > 0$. Now, let $x > \max\{n_0, \delta^{1/(1+\epsilon)}\}$. It then holds $1 - \delta(x + pi)^{-(1+\epsilon)} > 0$ for $i = 0, 1, \cdots$. Therefore, it follows from (9) that

$$\prod_{i=0}^{n-1} [1 - \delta(x+pi)^{-(1+\epsilon)}] \le P_n(x)/f(x) \le \prod_{i=0}^{n-1} [1 + \delta(x+pi)^{-(1+\epsilon)}]$$

since

$$\frac{P_n(x)}{f(x)} = \frac{f(x+pn)}{kf(x+p(n-1))} \frac{f(x+p(n-1))}{kf(x+p(n-2))} \cdots \frac{f(x+p)}{kf(x)}.$$

This implies the validity of (10). Now, it remains only to prove the uniqueness of F. Assume that $G : (0,\infty) \to [0,\infty)$ is another solution of the (p,k)-MP functional equation (1) and satisfies (10). Since both F and G are solutions of (1), it follows

$$\frac{F(x)}{G(x)} = \frac{F(x+pn)}{G(x+pn)} = \frac{F(x+pn)}{f(x+pn)} \frac{f(x+pn)}{G(x+pn)}$$

for any $x > 0$. Hence, we have

$$\frac{\alpha(x+pn)}{\beta(x+pn)} \le \frac{F(x)}{G(x)} \le \frac{\beta(x+pn)}{\alpha(x+pn)}$$

for all sufficiently large n. It is clear that the infinite products $\alpha(x)$ and $\beta(x)$ converges for all $x > 0$. Therefore, by using the relations

$$\alpha(x) = \lim_{n\to\infty} \alpha(x+pn) \lim_{n\to\infty} \prod_{i=0}^{n-1} [1-\delta(x+pi)^{-(1+\epsilon)}] = \lim_{n\to\infty} \alpha(x+pn)\alpha(x)$$

and

$$\begin{aligned}\beta(x) &= \lim_{n\to\infty} \beta(x+pn) \lim_{n\to\infty} \prod_{i=0}^{n-1} [1+\delta(x+pi)^{-(1+\epsilon)}] \\ &= \lim_{n\to\infty} \beta(x+pn)\beta(x),\end{aligned}$$

we conclude that $\alpha(x+pn) \to 1$ and $\beta(x+pn) \to 1$ as $n \to \infty$. Hence, it is obvious that $F(x) = G(x)$ holds true for all $x > 0$. □

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