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ON A FIBER SPACE WITH CONNECTED FIBERS

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ABSTRACT. Let $f : S \to C$ be a fiber space with connected fibers. We may have an information about a surface S from the fiber space structure. The result we have is $\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_F) \le \chi(\mathcal{O}_S)$.

Throughout this paper, we are working over the complex number field $\mathbb{C}$.

During investigating the rational map associated with the complete linear system on a surface S, we have often met a fiber space $f : S \to C$ with connected fibers. When we have this situation, we may get some information about a surface from the fiber space structure. The result we have is $\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_F) \le \chi(\mathcal{O}_S)$ (see Theorem A). This result can be obtained by combining the well-known theorem Hirzebruch-Riemann-Roch Theorem, the result about the direct image of a canonical sheaf (see Kollár [3]) and some result about the relative canonical sheaf $f_*K_{S/C}$ (see Kawamata [2] and Ueno [4]). For the detail matters about $f_*K_{S/C}$, see the proof of Theorem A. The aim of this note is to introduce the technique we have used and our result.

Now, we are going to set up our notations to use throughout in this note.

Let X be a smooth projective variety. Denote by K_X its canonical divisor. Denote the dimension of $H^i(X, \mathcal{O}_X(D))$ by $h^i(X, \mathcal{O}_X(D))$. Let's denote the genus of X by $p_g(X)$ and $h^1(X, \mathcal{O}_X)$ by $q(X)$ (or

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simply p_g and q unless there is any confusion). Denote by $\chi(\mathcal{O}_X)$ the Euler characteristic of X.

We need the following known theorems to show our result.

THEOREM 1. (Hirzebruch-Riemann-Roch theorem) *For a locally free sheaf $\mathcal{E}$ of rank r on X of dimension n, the Euler characteristic $\chi(\mathcal{E})$ of $\mathcal{E}$ is given as follows:*

$$\chi(\mathcal{E}) = deg(ch(\mathcal{E}) \cdot td(\mathcal{T}))_n,$$

where $ch(\mathcal{E})$ is the Chern character of $\mathcal{E}$, $td(\mathcal{T})$ is the Todd class of the tangent sheaf $\mathcal{T}$ of X and $(\ )_n$ means the component of degree n.

For the detail matters, see Hartshorne [1].

THEOREM 2. *Let S be a smooth projective surface and let C be a smooth projective curve. Let $f : S \to C$ be a surjective map with connected fibers. Then $R^1 f_* \mathcal{O}_S(K_S)$ is isomorphic to $\mathcal{O}_C(K_C)$.*

For a proof, see Kollár [3], Proposition 7.6.

Here is our result.

THEOREM A. *Let S be a smooth projective surface and let C be a smooth projective curve. Let $f : S \to C$ be a fiber space with connected fibers and let F be a general fiber of f. Then we have*

$$\chi(\mathcal{O}_F)\,\chi(\mathcal{O}_C) \le \chi(\mathcal{O}_S).$$

Proof. By the spectral sequence, we have

$$\begin{aligned} p_g &= h^0(S, \mathcal{O}_S(K_S)) = h^0(C, f_*\mathcal{O}_S(K_S)) \\ q &= h^1(S, \mathcal{O}_S(K_S)) = h^1(C, f_*\mathcal{O}_S(K_S)) + h^0(C, R^1 f_*\mathcal{O}_S(K_S)). \end{aligned}$$

By Theorem 2, $R^1 f_* \mathcal{O}_S(K_S)$ is isomorphic to $\mathcal{O}_C(K_C)$. So the genus $p_g(C)$ of C is equal to $h^0(C, R^1 f_* \mathcal{O}_S(K_S))$.

It is known that $f_*\mathcal{O}_S(K_S)$ is a locally free sheaf of rank $p_g(F)$ on C (see Ueno [4]). By Theorem 1, the Euler characteristic $\chi(f_*\mathcal{O}_S(K_S))$ on C is given by

$$\begin{aligned} & h^0(C, f_*\mathcal{O}_S(K_S)) - h^1(C, f_*\mathcal{O}_S(K_S)) \\ & = \chi(f_*\mathcal{O}_S(K_S)) \\ & = \deg f_*\mathcal{O}_S(K_S) + p_g(F)(1 - p_g(C)). \end{aligned}$$

Hence we have that

$$\begin{aligned} p_g - q = & h^0(C, f_*\mathcal{O}_S(K_S)) - h^1(C, f_*\mathcal{O}_S(K_S)) - h^0(C, R^1 f_*\mathcal{O}_S(K_S)) \\ = & \deg f_*\mathcal{O}_S(K_S) + p_g(F)(1 - p_g(C)) - p_g(C) \\ = & \deg f_* K_{S/C} + p_g(F)(p_g(C) - 1) - p_g(C), \end{aligned}$$

where $f_*K_{S/C} = f_*(\mathcal{O}_S(K_S) \otimes f^*\mathcal{O}_C(K_C)^{-1})$. Since $f_*K_{S/C}$ is semi-positive (for detail matters, see Kawamata [2], Theorem 6 or Ueno [4]), its degree is nonnegative. Hence we have

$$\begin{aligned} \chi(\mathcal{O}_S) & = p_g - q + 1 \\ & = \deg f_* K_{S/C} + p_g(F)(p_g(C) - 1) - p_g(C) + 1 \\ & = \deg f_* K_{S/C} + (p_g(F) - 1)(p_g(C) - 1). \end{aligned}$$

Since the first term is nonnegative, we have

$$\chi(\mathcal{O}_C)\,\chi(\mathcal{O}_F) \leq \chi(\mathcal{O}_S). \qquad \square$$

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